

BRAID GROUPS OF IMPRIMITIVE COMPLEX REFLECTION GROUPS

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ABSTRACT. We obtain new presentations for the imprimitive complex reflection groups of type (de, e, r) and their braid groups $B(de, e, r)$ for $d, r \geq 2$. Diagrams for these presentations are proposed. The presentations have much in common with Coxeter presentations of real reflection groups. They are positive and homogeneous, and give rise to quasi-Garside structures. Diagram automorphisms correspond to group automorphisms. The new presentation shows how the braid group $B(de, e, r)$ is a semidirect product of the braid group of affine type $\tilde{\mathbf{A}}_{r-1}$ and an infinite cyclic group. Elements of $B(de, e, r)$ are visualized as geometric braids on $r + 1$ strings whose first string is pure and whose winding number is a multiple of e . We classify periodic elements, and show that the roots are unique up to conjugacy and that the braid group $B(de, e, r)$ is strongly translation discrete.

Keywords: Complex reflection group; braid group; Garside group; periodic element; translation number.

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1. INTRODUCTION

1.1. Reflection groups and braid groups. A complex reflection group G on a finite dimensional complex vector space V is a subgroup of $\mathrm{GL}(V)$ generated by complex reflections—nontrivial elements that fix a complex hyperplane in V pointwise. Finite (irreducible) complex reflection groups were classified by Shephard and Todd [ST54]:

- (i) a general infinite family $G(de, e, r)$ for positive integral parameters d, e, r ;
- (ii) 34 exceptions, labeled G_4, G_5, \dots, G_{37} .

For the presentations of the above groups, see [BMR98].

Finite complex reflection groups are divided into two main classes: primitive and imprimitive. The general infinite family $G(de, e, r)$ are imprimitive except $G(1, 1, r)$ and $G(de, e, 1)$. ($G(1, 1, r)$ is the symmetric group of degree r and $G(de, e, 1)$ is the cyclic group of order d .) The exceptional groups G_4, G_5, \dots, G_{37} are primitive.

The complex reflection group of type (de, e, r) is defined as

$$G(de, e, r) = \left\{ \begin{array}{l} r \times r \text{ monomial matrices} \\ (x_{ij}) \text{ over } \{0\} \cup \mu_{de} \end{array} \mid \prod_{x_{ij} \neq 0} x_{ij}^d = 1 \right\},$$

where μ_{de} is the set of de -th roots of unity. Special cases of $G(de, e, r)$ are isomorphic to real reflection groups: $G(1, 1, r) \cong G(\mathbf{A}_{r-1})$, $G(2, 1, r) \cong G(\mathbf{B}_r)$, $G(2, 2, r) \cong G(\mathbf{D}_r)$ and $G(e, e, 2) \cong G(\mathbf{I}_2(e))$, where $G(W)$ denotes the Coxeter group of type W . For all the other parameters, $G(de, e, r)$ has no real structure.

The braid group of a complex reflection group is defined as the fundamental group of the regular orbits. For these braid groups, the presentations and the centers are shown in [BMR98, BDM02, BM04, Bes06b].

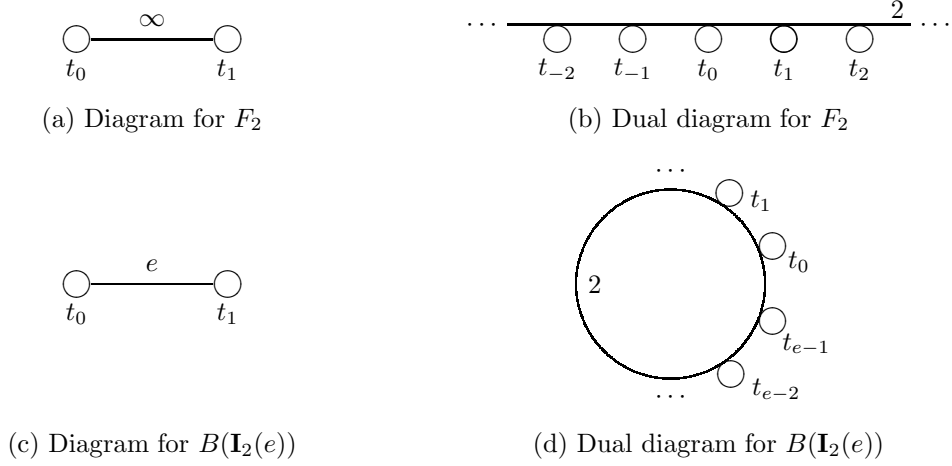
The braid groups $B(de, e, r)$ of the complex reflection groups $G(de, e, r)$ are divided into two cases: $d = 1$ and $d \geq 2$. For any $d, d' \geq 2$, $B(de, e, r) = B(d'e, e, r) \neq B(e, e, r)$ [BMR98]. It was shown in [BC06, CP11] that the braid groups $B(e, e, r)$ are Garside groups.

1.2. Outline and main results. In this paper, we propose new presentations and diagrams for the imprimitive reflection groups $G(de, e, r)$ and their braid groups $B(de, e, r)$ for $d, r \geq 2$ and $e \geq 1$. These presentations have much in common with Coxeter presentations of real reflection groups. They are positive and homogeneous, and give rise to quasi-Garside structures. Diagram automorphisms correspond to group automorphisms.

To motivate our approach, we review in §2 the presentations for the free group F_2 and the braid groups $B(\mathbf{I}_2(e))$ and $B(e, e, r)$.

As a generic version of the imprimitive reflection groups $G(e, e, r)$, Shi [Shi02] introduced the complex reflection group $G(\infty, \infty, r)$ and showed that $G(\infty, \infty, r)$ is isomorphic to the affine reflection group of type $\tilde{\mathbf{A}}_{r-1}$. In §3, we propose new presentations for $G(\infty, \infty, r)$ and its braid group $B(\infty, \infty, r)$. We show how the braid group $B(de, e, r)$ is a semidirect product of $B(\infty, \infty, r)$ and an infinite cyclic group, and then show how $G(de, e, r)$ is a semidirect product of $G(de, de, r)$ and a cyclic group of order d . The new presentations give rise to quasi-Garside structures on the braid groups $B(\infty, \infty, r)$ and $B(de, e, r)$.

In §4, we explore some properties of $B(de, e, r)$. Elements of $B(de, e, r)$ will be interpreted as geometric braids on $r + 1$ strings whose first string is pure and whose winding number around the


 FIGURE 1. Diagrams for F_2 and $B(\mathbf{I}_2(e))$

first string is a multiple of e . Using this interpretation, we show that the k -th root of an element of $B(de, e, r)$, if exists, is unique up to conjugacy for any nonzero integer k , show that $B(de, e, r)$ is strongly translation discrete, and classify periodic elements of $B(de, e, r)$.

The braid group $B(\infty, \infty, r)$ is isomorphic to $B(\tilde{\mathbf{A}}_{r-1})$. In §5, we propose an $\tilde{\mathbf{A}}$ -type presentation for $B(de, e, r)$ which is also positive and homogeneous.

Notations. Denote by $\langle b_1 b_2 \cdots b_l \rangle^k$ the cyclic product $b_1 b_2 \cdots b_l b_1 b_2 \cdots$ with k factors. For example, $\langle b_1 b_2 b_3 \rangle^2 = b_1 b_2 b_3 b_1 b_2 b_3$ and $\langle b_1 b_2 b_3 \rangle^5 = b_1 b_2 b_3 b_1 b_2 b_3 b_1 b_2 b_3$. If w is the word $b_1 b_2 \cdots b_l$, then we will write $\langle w \rangle^k$ as a shorthand for $\langle b_1 b_2 \cdots b_l \rangle^k$.

2. PRELIMINARY MATERIAL

2.1. The free group F_2 and the braid group of type $\mathbf{I}_2(e)$. Here we review the presentations of the free group F_2 and the braid group of type $\mathbf{I}_2(e)$. These presentations are not necessary for the work of this paper, but they will give an intuition about our approach to the braid group $B(de, e, r)$.

2.1.1. Presentations for the free group F_2 . Consider the following two presentations for the free group F_2 :

$$\begin{aligned} F_2 &= \langle t_0, t_1 \mid \quad \rangle; \\ F_2 &= \langle t_i, i \in \mathbb{Z} \mid t_i t_{i-1} = t_j t_{j-1} \text{ for all } i, j \in \mathbb{Z} \rangle. \end{aligned}$$

The second presentation is obtained from the first one by adding new generators t_i for $i \in \mathbb{Z} \setminus \{0, 1\}$ together with defining relations $\cdots = t_2 t_1 = t_1 t_0 = t_0 t_{-1} = \cdots$. These presentations can be described as in Figure 1(a,b). The first diagram is also known as $\tilde{\mathbf{A}}_1$. In the diagram for the second presentation, countably many nodes are tangent to a line labeled 2, which means the relation $t_i t_{i-1} = t_j t_{j-1}$ for $i, j \in \mathbb{Z}$. It is known that the second presentation gives rise to a quasi-Garside structure on F_2 [Bes06a, DDGKM14]. One may regard it as a *dual* presentation of the first one.

2.1.2. *Presentations for the braid group $B(\mathbf{I}_2(e))$.* The following are well-known presentations for the braid group of the dihedral group on $2e$ elements, denoted $B(\mathbf{I}_2(e))$:

$$\begin{aligned} B(\mathbf{I}_2(e)) &= \langle t_0, t_1 \mid \langle t_0 t_1 \rangle^e = \langle t_1 t_0 \rangle^e \rangle; \\ B(\mathbf{I}_2(e)) &= \langle t_0, t_1, \dots, t_{e-1} \mid t_1 t_0 = t_2 t_1 = \dots = t_{e-1} t_{e-2} = t_0 t_{e-1} \rangle. \end{aligned}$$

See Figure 1(c,d). It is easy to see that the above two presentations are equivalent. The first presentation is usually referred as the *classical* presentation, and the second as the *dual* presentation. Both presentations give rise to Garside structures.

The free group F_2 can be considered as a version of $B(\mathbf{I}_2(e))$ where e is replaced by ∞ .

2.2. **The braid group $B(e, e, r)$.** Broué, Malle and Rouquier [BMR98] obtained the following presentation for the braid group $B(e, e, r)$ for $e, r \geq 2$:

- Generators: $\{t_0, t_1\} \cup S$ where $S = \{s_3, \dots, s_r\}$;
- Relations: the usual braid relations on S , along with
 - (P₁) $\langle t_1 t_0 \rangle^e = \langle t_0 t_1 \rangle^e$,
 - (P₂) $s_3 t_i s_3 = t_i s_3 t_i$ for $i = 0, 1$,
 - (P₃) $s_j t_i = t_i s_j$ for $i = 0, 1$ and $4 \leq j \leq r$,
 - (P₄) $s_3(t_1 t_0) s_3(t_1 t_0) = (t_1 t_0) s_3(t_1 t_0) s_3$.

Furthermore, a presentation for the reflection group $G(e, e, r)$ is obtained by adding the relation $a^2 = 1$ for all generators a , and the generators are then all reflections.

The presentation is usually illustrated by the diagram shown in Figure 2(a). The diagram is to be read as a Coxeter graph for the real reflection group case: when nodes a and b are joined by an edge labeled e , there is a relation $\langle ab \rangle^e = \langle ba \rangle^e$; when nodes a and b are joined by an unlabelled edge, there is a relation $aba = bab$; when two nodes a and b are not connected by an edge, there is a relation $ab = ba$; the double line “ \equiv ” between the node s_3 and the edge connecting the nodes t_1 and t_0 indicates the relation $s_3(t_1 t_0) s_3(t_1 t_0) = (t_1 t_0) s_3(t_1 t_0) s_3$.

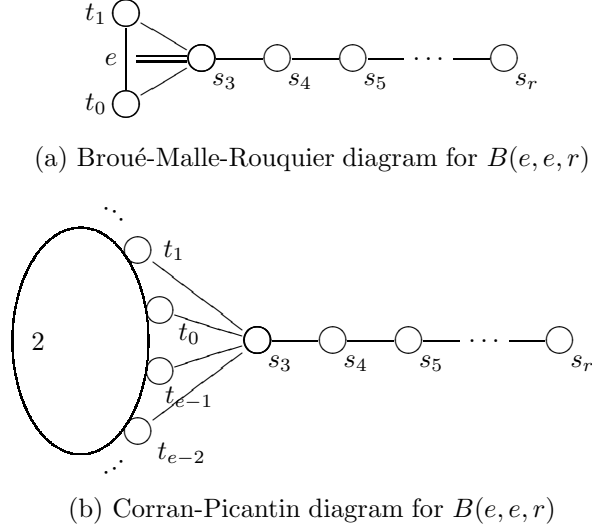
Setting $r = 2$ results in the classical presentation of type $\mathbf{I}_2(e)$, and the subpresentation on the generators s_3, \dots, s_r is the classical presentation of type \mathbf{A}_{r-2} (see next subsection). Indeed, the subpresentation on the generators t_i, s_3, \dots, s_r is the classical presentation of type \mathbf{A}_{r-1} .

2.2.1. *Corran-Picantin presentation of $B(e, e, r)$.* The first author and Picantin [CP11] obtained the following presentation for the braid group $B(e, e, r)$, the generators of which are *braid reflections* and which gives rise to a Garside structure.

Theorem 2.1 ([CP11]). *The braid group $B(e, e, r)$ for $e, r \geq 2$ has the following presentation:*

- Generators: $T_e \cup S$ where $T_e = \{t_i \mid i \in \mathbb{Z}/e\}$ and $S = \{s_3, \dots, s_r\}$;
- Relations: the usual braid relations on S , along with
 - (Q₁) $t_i t_{i-1} = t_j t_{j-1}$ for $i, j \in \mathbb{Z}/e$,
 - (Q₂) $s_3 t_i s_3 = t_i s_3 t_i$ for $i \in \mathbb{Z}/e$,
 - (Q₃) $s_j t_i = t_i s_j$ for $i \in \mathbb{Z}/e$ and $4 \leq j \leq r$.

Furthermore, adding the relations $a^2 = 1$ for all generators a gives a presentation of the imprimitive reflection group $G(e, e, r)$, where the generators are all reflections.


 FIGURE 2. Diagrams for $B(e, e, r)$

Denote by $\bar{\cdot}$ the natural map $B(e, e, r) \rightarrow G(e, e, r)$. The generating reflections of $G(e, e, r)$ in this new presentation are the following $r \times r$ matrices:

$$\bar{t}_i = \left(\begin{array}{cc|c} 0 & \zeta_e^{-i} & 0 \\ \zeta_e^i & 0 & 0 \\ \hline 0 & & I_{r-2} \end{array} \right) \quad \text{and} \quad \bar{s}_j = \text{permutation matrix of } (j-1 \ j),$$

where ζ_e is a primitive e -th root of unity.

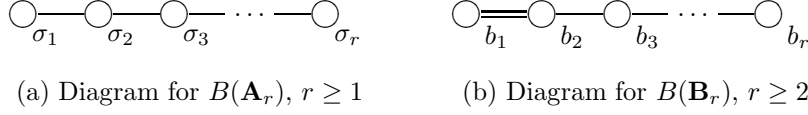
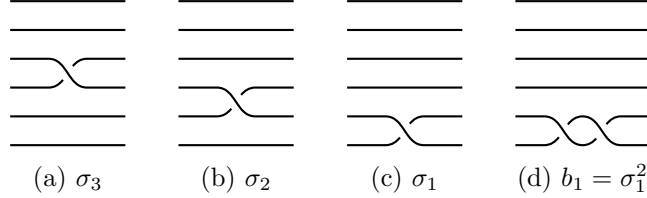
2.2.2. Diagram of type (e, e, r) . The diagram shown in Figure 2(b) was proposed in [CP11], as a type (e, e, r) -analogy to the Coxeter graphs for the real reflection group case. In the diagram, there are e nodes labeled t_i , $i \in \mathbb{Z}/e$, that are tangent to a circle labeled 2. Whenever two nodes a and b are tangent to the circle, there is a relation of the form $aa' = bb'$ where a' and b' are the nodes immediately preceding a and b respectively on the circle. If two nodes a and b are neither connected by an edge nor tangent to the circle, then there is a relation of the form $ab = ba$.

Naively, this appears like the *dual* presentation of $B(\mathbf{I}_2(e))$ (on the generators t_i) combined with the *classical* presentation of $B(\mathbf{A}_{r-2})$ (on the generators s_i).

The group of graph automorphisms of the diagram is the cyclic group of order e , which may be generated by the automorphism τ which rotates the circle in the positive direction by a turn of $2\pi/e$. This sends the node t_i to the node t_{i+1} for $i \in \mathbb{Z}/e$, and fixes the nodes s_j for $3 \leq j \leq r$. By the symmetry of the presentation, these diagram automorphisms give rise to automorphisms of the braid group $B(e, e, r)$ as well as of the reflection group $G(e, e, r)$. These automorphisms send (braid) reflections to (braid) reflections.

2.2.3. Maps between the groups $B(e, e, r)$ for different values of e . Consider a sequence of natural numbers $\{e_i\}_{i \geq 0}$ such that $e_0 = 1$ and e_i divides e_{i+1} for each $i \geq 0$. For each i , define an epimorphism

$$\nu_{e_i}^{e_{i+1}} : B(e_{i+1}, e_{i+1}, r) \twoheadrightarrow B(e_i, e_i, r)$$

FIGURE 3. Coxeter graphs of $B(\mathbf{A}_r)$ and $B(\mathbf{B}_r)$ FIGURE 4. Braid pictures for σ_3 , σ_2 , σ_1 and $b_1 = \sigma_1^2$ in B_6

by $t_k \mapsto t_{k \bmod e_i}$ and $s_j \mapsto s_j$. Then

$$\nu_{e_i}^{e_{i+1}} \circ \nu_{e_{i+1}}^{e_{i+2}} = \nu_{e_i}^{e_{i+2}} \quad \text{for all } i \geq 0.$$

2.3. Braid groups of types \mathbf{A} , \mathbf{B} and $\tilde{\mathbf{A}}$ and geometric braids. The group $B(1, 1, r+1)$ for $r \geq 1$ is precisely the Artin braid group B_{r+1} on $r+1$ strings—also known as the braid group of type \mathbf{A}_r , denoted $B(\mathbf{A}_r)$ —and possesses the following presentation.

$$B(\mathbf{A}_r) = B_{r+1} = \left\langle \sigma_1, \dots, \sigma_r \left| \begin{array}{ll} \sigma_i \sigma_j = \sigma_j \sigma_i & \text{for } |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} & \text{for } i = 1, \dots, r-1 \end{array} \right. \right\rangle$$

The group $B(2, 1, r)$ (or indeed $B(d, 1, r)$ for any $d \geq 2$) for $r \geq 2$ is usually called the braid group of type \mathbf{B}_r , denoted $B(\mathbf{B}_r)$, and has the following presentation.

$$(1) \quad B(\mathbf{B}_r) = \left\langle b_1, \dots, b_r \left| \begin{array}{ll} b_i b_j = b_j b_i & \text{for } |i-j| > 1 \\ b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} & \text{for } 1 < i < r \\ b_1 b_2 b_1 b_2 = b_2 b_1 b_2 b_1 \end{array} \right. \right\rangle$$

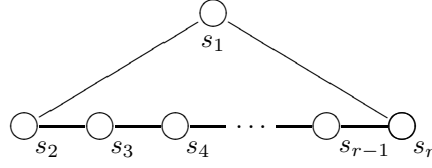
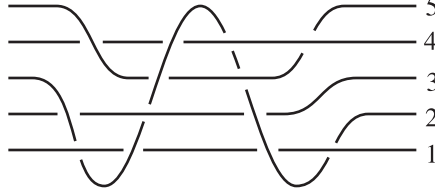
See Figure 3 for the diagrams for the above presentations. The double edge between b_1 and b_2 encodes the relation $b_1 b_2 b_1 b_2 = b_2 b_1 b_2 b_1$. The braid group $B(\mathbf{B}_r)$ is a subgroup of B_{r+1} of index $r+1$ under the identification $b_1 = \sigma_1^2$ and $b_i = \sigma_i$ for $i = 2, \dots, r$. See Figure 4.

The braid group of type $\tilde{\mathbf{A}}_{r-1}$ for $r \geq 3$, denoted $B(\tilde{\mathbf{A}}_{r-1})$, is usually described by the Coxeter graph in Figure 5. This diagram defines a presentation

$$(2) \quad B(\tilde{\mathbf{A}}_{r-1}) = \left\langle s_1, \dots, s_r \left| \begin{array}{ll} s_i s_j = s_j s_i & \text{for } i-j \not\equiv \pm 1 \pmod r \\ s_i s_j s_i = s_j s_i s_j & \text{for } i-j \equiv \pm 1 \pmod r \end{array} \right. \right\rangle.$$

On adding the relations $s_i^2 = 1$ for all i , a presentation for the affine reflection group of type $\tilde{\mathbf{A}}_{r-1}$ is obtained, where the generators are (real affine) reflections.

Definition 2.2. (i) A braid $g \in B_{r+1}$ is said to be i -pure for $1 \leq i \leq r+1$ if $\pi_g(i) = i$, where π_g denotes the induced permutation of g . Let $B_{r+1,1}$ denote the subgroup of B_{r+1} consisting of 1-pure braids, hence it is generated by $\{\sigma_1^2, \sigma_2, \sigma_3, \dots, \sigma_r\}$.
(ii) Let P be a subset of $\{1, \dots, r+1\}$. An $(r+1)$ -braid g is said to be P -pure if g is i -pure for each $i \in P$. It is said to be P -straight if it is P -pure and it becomes trivial when we remove all the i -th strings for $i \notin P$.


 FIGURE 5. Coxeter graph of $B(\tilde{\mathbf{A}}_{r-1})$, $r \geq 3$

 FIGURE 6. This braid is $\{1, 4, 5\}$ -pure, $\{1, 4\}$ -straight and $\{1, 5\}$ -straight, and has winding number 0.

- (iii) The homomorphism $\text{wd} : B_{r+1,1} \rightarrow \mathbb{Z}$ measures the *winding number* around the first string of the other strings. In particular, $\text{wd}(\sigma_1^2) = 1$ and $\text{wd}(\sigma_i) = 0$ for all $2 \leq i \leq r$.

For example, the braid in Figure 6 is $\{1, 4, 5\}$ -pure, $\{1, 4\}$ -straight and $\{1, 5\}$ -straight, and has winding number 0. Notice that if $|P| = 1$, then a braid is P -pure if and only if it is P -straight, and that if $P = \{1, \dots, r+1\}$, then P -pure braids are nothing more than pure braids in the usual sense and the identity braid is the only P -straight braid.

It is well known that the braid group $B(\mathbf{B}_r)$ is isomorphic to the group of r -braids on an annulus and the braid group $B(\tilde{\mathbf{A}}_{r-1})$ is isomorphic to the subgroup consisting of such braids with zero winding number [Cri99, All02, CC05, BM07]. Equivalently, $B(\mathbf{B}_r)$ is isomorphic to the group of 1-pure $(r+1)$ -braids on a disk and $B(\tilde{\mathbf{A}}_{r-1})$ is isomorphic to the subgroup consisting of such braids with zero winding number around the first string. In this paper, we use the latter isomorphisms

$$B(\mathbf{B}_r) \cong B_{r+1,1},$$

$$B(\tilde{\mathbf{A}}_{r-1}) \cong \{g \in B_{r+1,1} \mid \text{wd}(g) = 0\}.$$

The embedding of $B(\tilde{\mathbf{A}}_{r-1})$ into $B(\mathbf{B}_r)$ can be made explicit in the following way. Consider a regular r -gon in the interior of a disk whose edges are labeled E_1, \dots, E_r in clockwise order as in Figure 7(a). Suppose that one puncture is at the center and r punctures are on the vertices of the r -gon. As mapping classes, the generator s_i of $B(\tilde{\mathbf{A}}_{r-1})$ is represented by a positive half Dehn twist along E_i . The configuration in Figure 7(a) is equivalent to that in Figure 7(b), from which the generators s_i are expressed as words in σ_j 's as follows.

$$s_1 = (\sigma_r \cdots \sigma_3)(\sigma_1^{-2} \sigma_2 \sigma_1^2)(\sigma_3^{-1} \cdots \sigma_r^{-1}),$$

$$s_j = \sigma_j \quad \text{for } 2 \leq j \leq r.$$

See Figure 7(c) for the braid picture of the generator s_1 .

In the same way that the free group on two generators can be considered as the braid group of a dihedral group $\mathbf{I}_2(\infty)$, the parameter e of the braid group $B(e, e, r)$ can be set to infinity to

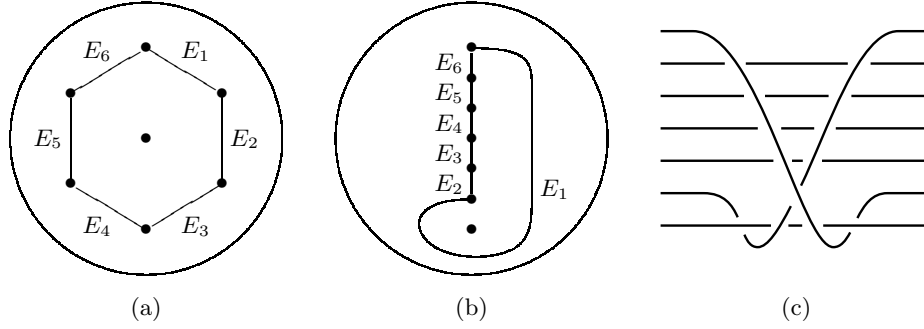


FIGURE 7. (a) and (b) are equivalent configurations of punctures and arcs. As mapping classes, the generator s_i of $B(\tilde{\mathbf{A}}_{r-1})$ is represented by a positive half Dehn twist along E_i . (c) shows a braid picture for the generator s_1 , which is a 1-pure braid with winding number zero.

obtain a group $B(\infty, \infty, r)$. This turns out to be isomorphic to the braid group of type $\tilde{\mathbf{A}}_{r-1}$ —see the following section and §5 for an extended discussion.

3. NEW PRESENTATIONS FOR THE BRAID GROUPS $B(\infty, \infty, r)$ AND $B(de, e, r)$

By suppressing the relation (P_1) in the Broué-Malle-Rouquier presentation of $B(e, e, r)$, we obtain a group which we will denote by $B(\infty, \infty, r)$. In other words, $B(\infty, \infty, r)$, $r \geq 2$, has the following presentation (see Figure 8(a) for the diagram for this presentation):

- Generators: $\{t_0, t_1\} \cup S$ where $S = \{s_3, \dots, s_r\}$;
- Relations: the usual braid relations on S , along with
 - $(P_2) \quad s_3 t_i s_3 = t_i s_3 t_i \quad \text{for } i = 0, 1,$
 - $(P_3) \quad s_j t_i = t_i s_j \quad \text{for } i = 0, 1 \text{ and } 4 \leq j \leq r,$
 - $(P_4) \quad s_3(t_1 t_0) s_3(t_1 t_0) = (t_1 t_0) s_3(t_1 t_0) s_3.$

This group was first considered by Shi [Shi02] as a generic version of the groups $B(e, e, r)$. He observed that on adding the relations $a^2 = 1$ for all the generators a , a presentation was obtained for a group denoted $G(\infty, \infty, r)$. Denote by $\bar{}$ the natural map $B(\infty, \infty, r) \rightarrow G(\infty, \infty, r)$. The generating reflections in the presentation of $G(\infty, \infty, r)$ are the $r \times r$ matrices

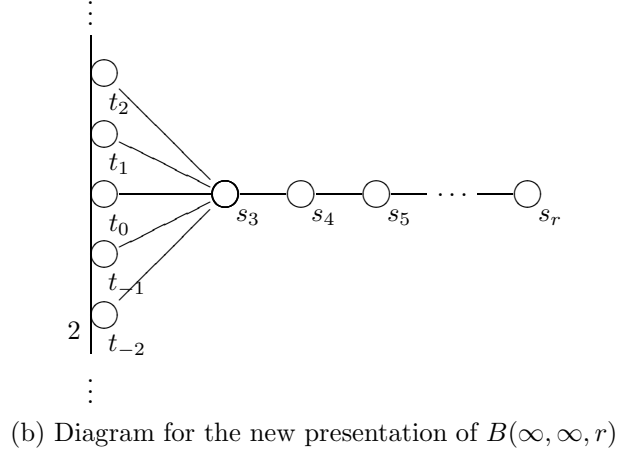
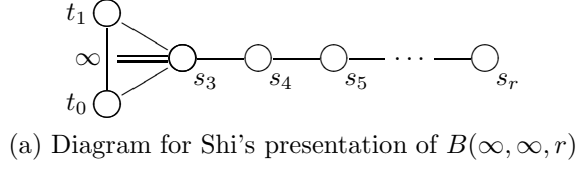
$$\bar{t}_i = \left(\begin{array}{cc|c} 0 & x^{-i} & 0 \\ x^i & 0 & 0 \\ \hline 0 & & I_{r-2} \end{array} \right) \quad \text{and} \quad \bar{s}_j = \text{permutation matrix of } (j-1 \ j),$$

where x is a transcendental number. The matrices are complex reflections of order 2.

3.1. New presentation of $B(\infty, \infty, r)$. For the braid group $B(\infty, \infty, r)$, we introduce a new presentation, analogous to the Corran-Picantin presentation of $B(e, e, r)$.

Theorem 3.1. *The braid group $B(\infty, \infty, r)$ for $r \geq 2$ has the following presentation:*

- Generators: $T \cup S$ where $T = \{t_i \mid i \in \mathbb{Z}\}$ and $S = \{s_3, \dots, s_r\}$;


 FIGURE 8. Diagrams for $B(\infty, \infty, r)$

- *Relations: the usual braid relations on S , along with*
 - (Q_1) $t_i t_{i-1} = t_j t_{j-1}$ for $i, j \in \mathbb{Z}$,
 - (Q_2) $s_3 t_i s_3 = t_i s_3 t_i$ for $i \in \mathbb{Z}$,
 - (Q_3) $s_j t_i = t_i s_j$ for $i \in \mathbb{Z}$ and $4 \leq j \leq r$.

Furthermore, adding the relations $a^2 = 1$ for all generators a gives a presentation of the group $G(\infty, \infty, r)$, where the generators are all reflections.

The proof of the above theorem is the same as that for $B(e, e, r)$ in [CP11]. However, we give the proof in §3.1.3 for completeness.

3.1.1. *Diagram of type (∞, ∞, r) .* In the obvious generalization of the type (e, e, r) case, we propose the diagram shown in Figure 8(b) as a type (∞, ∞, r) diagram. Notice that the diagram for the new presentation is obtained from Shi's diagram by changing $\begin{array}{c} \circ \\ \text{---} \infty \text{---} \circ \\ t_0 \quad t_1 \end{array}$ to the dual diagram for F_2 .

Let τ denote the graph automorphism

$$t_i \mapsto t_{i+1} \quad \text{for } i \in \mathbb{Z}, \quad s_j \mapsto s_j \quad \text{for } 3 \leq j \leq r.$$

This gives rise to automorphisms of $B(\infty, \infty, r)$ and $G(\infty, \infty, r)$ which send (braid) reflections to (braid) reflections.

3.1.2. *Maps between $B(e, e, r)$ and $B(\infty, \infty, r)$.* Let $\nu_e : B(\infty, \infty, r) \rightarrow B(e, e, r)$ be the epimorphism which sends t_i to $t_{i \bmod e}$ for $i \in \mathbb{Z}$ and sends s_j to s_j for $3 \leq j \leq r$. Once again, consider a sequence of natural numbers e_i such that $e_0 = 1$ and e_i divides e_{i+1} for each $i \geq 0$. Then

$$\nu_{e_i}^{e_{i+1}} \circ \nu_{e_{i+1}} = \nu_{e_i} \quad \text{and} \quad \nu_{e_i}^{e_{i+1}} \circ \nu_{e_{i+1}}^{e_{i+2}} = \nu_{e_i}^{e_{i+2}} \quad \text{for all } i \geq 0.$$

The inverse limit of the sequence

$$\cdots \longrightarrow B(e_{i+1}, e_{i+1}, r) \xrightarrow{\nu_{e_i}^{e_{i+1}}} B(e_i, e_i, r) \xrightarrow{\nu_{e_{i-1}}^{e_i}} \cdots \xrightarrow{\nu_{e_1}^{e_2}} B(e_1, e_1, r) \xrightarrow{\nu_1^{e_1}} B(1, 1, r)$$

is however *not* $B(\infty, \infty, r)$, but rather a group we denote by $B(\hat{\mathbb{Z}}, \hat{\mathbb{Z}}, r)$, where $\hat{\mathbb{Z}}$ is the profinite completion of \mathbb{Z} [Shi02].

3.1.3. *Proof of Theorem 3.1.* For completeness, we include a proof of Theorem 3.1. Add new generators $\{t_i \mid i \in \mathbb{Z} \setminus \{0, 1\}\}$ to Shi's presentation which are defined inductively by

$$t_i = \begin{cases} t_{i-1}t_{i-2}t_{i-1}^{-1} & \text{for } i \geq 2, \\ t_{i+1}^{-1}t_{i+2}t_{i+1} & \text{for } i \leq -1. \end{cases}$$

The above relations are the same as $t_it_{i-1} = t_{i-1}t_{i-2}$ for all $i \in \mathbb{Z}$, which is the same as the relation (Q_1) . Therefore $B(\infty, \infty, r)$ has the following presentation:

- Generators: $T \cup S$ where $T = \{t_i \mid i \in \mathbb{Z}\}$ and $S = \{s_3, \dots, s_r\}$;
- Relations: the usual braid relations on S , along with
 - (Q_1) $t_it_{i-1} = t_jt_{j-1}$ for $i, j \in \mathbb{Z}$,
 - (P_2) $s_3t_is_3 = t_is_3t_i$ for $i = 0, 1$,
 - (P_3) $s_jt_i = t_is_j$ for $i = 0, 1$ and $4 \leq j \leq r$,
 - (P_4) $s_3(t_1t_0)s_3(t_1t_0) = (t_1t_0)s_3(t_1t_0)s_3$.

Claim 1. Assuming (Q_1) , the relation (Q_3) is equivalent to (P_3) .

Proof of Claim 1. (P_3) is a special case of (Q_3) . Assuming (Q_1) , every t_i is represented by a word in $\{t_0, t_1\}$: for $m \in \mathbb{Z}$ and $k = 0, 1$,

$$t_{2m+k} = (t_1t_0)^m t_k (t_1t_0)^{-m}.$$

If we assume (P_3) , then s_j ($4 \leq j \leq r$) commutes with t_0 and t_1 , hence s_j commutes with t_i for any $i \in \mathbb{Z}$, which is the relation (Q_3) . Therefore (P_3) implies (Q_3) . \square

Claim 2. Assuming (Q_1) , the relation (Q_2) is equivalent to $(P_2) + (P_4)$.

Proof of Claim 2. Suppose that (Q_2) holds, that is, $s_3t_is_3 = t_is_3t_i$ for all $i \in \mathbb{Z}$. Notice that (P_2) is a special case of (Q_2) . The following formula shows that (P_4) also holds, where relations are applied to the underlined subwords.

$$\begin{aligned} s_3 \underline{t_1 t_0} s_3 t_1 t_0 &\stackrel{Q_1}{=} s_3 t_2 \underline{t_1 s_3 t_1} t_0 \stackrel{Q_2}{=} \underline{s_3 t_2 s_3} t_1 s_3 t_0 \stackrel{Q_2}{=} t_2 s_3 \underline{t_2 t_1} s_3 t_0 \\ &\stackrel{Q_1}{=} t_2 s_3 t_1 \underline{t_0 s_3 t_0} \stackrel{Q_2}{=} t_2 \underline{s_3 t_1 s_3} t_0 s_3 \stackrel{Q_2}{=} \underline{t_2 t_1} s_3 t_1 t_0 s_3 \stackrel{Q_1}{=} t_1 t_0 s_3 t_1 t_0 s_3. \end{aligned}$$

Conversely, suppose that both (P_2) and (P_4) hold. By (P_2) , we know that (Q_2) holds for $i = 0, 1$. Assume that (Q_2) holds for $i = k, k+1$, which we denote by $(Q_{2,k})$ and $(Q_{2,k+1})$, respectively. Since

$$\begin{aligned} s_3 t_{k+2} \underline{s_3 t_{k+1} s_3} t_k &\stackrel{Q_{2,k+1}}{=} s_3 \underline{t_{k+2} t_{k+1}} s_3 \underline{t_{k+1} t_k} \stackrel{Q_1}{=} s_3 t_1 t_0 s_3 t_1 t_0 \stackrel{P_4}{=} \underline{t_1 t_0} s_3 \underline{t_1 t_0} s_3 \\ &\stackrel{Q_1}{=} t_{k+2} \underline{t_{k+1} s_3 t_{k+1}} t_k s_3 \stackrel{Q_{2,k+1}}{=} t_{k+2} s_3 t_{k+1} \underline{s_3 t_k s_3} \\ &\stackrel{Q_{2,k}}{=} t_{k+2} s_3 \underline{t_{k+1} t_k} s_3 t_k \stackrel{Q_1}{=} t_{k+2} s_3 t_{k+2} t_{k+1} s_3 t_k, \end{aligned}$$

we have $s_3 t_{k+2} s_3 = t_{k+2} s_3 t_{k+2}$ by canceling $t_{k+1} s_3 t_k$ from the right. Hence (Q_2) holds for $i = k+2$. Similarly, we can show that (Q_2) holds also for $i = k-1$. By induction, we conclude that (Q_2) holds for all $i \in \mathbb{Z}$. \square

From Claims 1 and 2, the new presentation for $B(\infty, \infty, r)$ is correct.

The new presentation has the same generators as the original, as well as some conjugates of the originals. Since it is the case for Shi's presentation, adding the relations $a^2 = 1$ for all generators a in the new presentation gives a presentation of the reflection group $G(\infty, \infty, r)$. Since conjugates of reflections are reflections, the generators of this presentation are all reflections.

3.2. New presentation of $B(de, e, r)$. Broué, Malle and Rouquier [BMR98] introduced the following presentation of $B(de, e, r)$:

- Generators: $\{z\} \cup \{t_0, t_1\} \cup S$ where $S = \{s_j \mid 3 \leq j \leq r\}$;
- Relations: the usual braid relations on S , along with
 - (R_1) $zt_1 t_0 = t_1 t_0 z$,
 - (R_2) $z\langle t_1 t_0 \rangle^e = t_0 z\langle t_1 t_0 \rangle^{e-1}$,
 - (R_3) $zs_j = s_j z$ for $3 \leq j \leq r$,
 - (R_4) $s_3 t_i s_3 = t_i s_3 t_i$ for $i = 0, 1$,
 - (R_5) $s_3(t_1 t_0) s_3(t_1 t_0) = (t_1 t_0) s_3(t_1 t_0) s_3$,
 - (R_6) $s_j t_i = t_i s_j$ for $i = 0, 1$ and $4 \leq j \leq r$.

Furthermore, a presentation for the reflection group $G(de, e, r)$ is obtained by adding the relations $z^d = 1$, $t_0^2 = t_1^2 = 1$ and $s_j^2 = 1$ for $3 \leq j \leq r$, and the generators are then all reflections.

This presentation is usually illustrated by the diagram in Figure 9(a) (note the similarity to the diagram for $B(e, e, r)$ in Figure 2(a)). Adding the relation $z = 1$ to the above presentation gives the BMR presentation for $B(e, e, r)$, thus defining an epimorphism from $B(de, e, r)$ to $B(e, e, r)$.

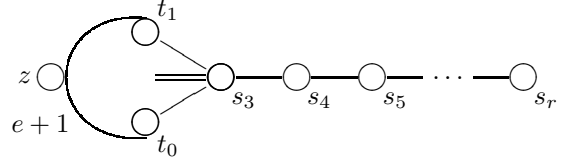
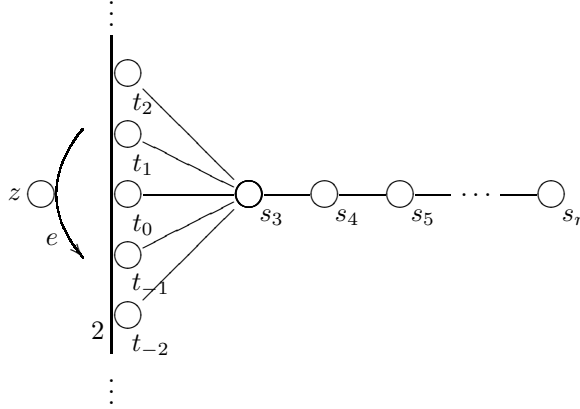
In the case $e = 1$, type $(de, e, r) = (d, 1, r)$ is precisely type \mathbf{B}_r . The above presentation is claimed in [BMR98] as valid for $d, e, r \geq 2$ (probably to avoid doubling up for the type \mathbf{B}_r presentation). However it is indeed valid in the $e = 1$ case as well. In this case, the BMR relation (R_2) becomes $t_1 = z^{-1} t_0 z$, hence t_1 is a superfluous generator. If we remove t_1 from the set of generators and replace every occurrence of t_1 in the defining relations with $z^{-1} t_0 z$, then the BMR presentation is reduced to the presentation of type \mathbf{B}_r , under the correspondence $z \mapsto b_1$, $t_0 \mapsto b_2$ and $s_i \mapsto b_i$ for $3 \leq i \leq r$.

In this article, when we speak of the BMR-presentation, it will be implicit that $d, r \geq 2$ and $e \geq 1$.

3.2.1. Semidirect product with $B(\infty, \infty, r)$. We propose a new presentation for $B(de, e, r)$ which makes clear the decomposition of $B(de, e, r)$ as a semidirect product of $B(\infty, \infty, r)$ and an infinite cyclic group. The theorem will be proved in §3.2.4.

Theorem 3.2. *The braid group $B(de, e, r)$ for $d, r \geq 2$ and $e \geq 1$ has the following presentation:*

- Generators: $\{z\} \cup T \cup S$ where $T = \{t_i \mid i \in \mathbb{Z}\}$ and $S = \{s_j \mid 3 \leq j \leq r\}$;

(a) Broué-Malle-Rouquier diagram for $B(de, e, r)$ (b) Diagram for the new presentation of $B(de, e, r)$ FIGURE 9. Diagrams for $B(de, e, r)$

- *Relations: the usual braid relations on S , along with*

$$(Q_1) \quad t_i t_{i-1} = t_j t_{j-1} \quad \text{for } i, j \in \mathbb{Z},$$

$$(Q_2) \quad s_3 t_i s_3 = t_i s_3 t_i \quad \text{for } i \in \mathbb{Z},$$

$$(Q_3) \quad s_j t_i = t_i s_j \quad \text{for } i \in \mathbb{Z} \text{ and } 4 \leq j \leq r,$$

$$(Q_4) \quad z t_i = t_{i-e} z \quad \text{for } i \in \mathbb{Z},$$

$$(Q_5) \quad z s_j = s_j z \quad \text{for } 3 \leq j \leq r.$$

Let $C_\infty = \langle c \rangle$ be an infinite cyclic group acting on $B(\infty, \infty, r)$ as follows:

$$c \cdot t_i = t_{i-1} \quad \text{for } i \in \mathbb{Z}, \quad c \cdot s_j = s_j \quad \text{for } 3 \leq j \leq r.$$

Let $z = c^e$ and $C_\infty^e = \langle z \rangle$. Then the presentation of $B(de, e, r)$ in Theorem 3.2 can be considered as a presentation of $C_\infty^e \ltimes B(\infty, \infty, r)$: the relations (Q_1) , (Q_2) and (Q_3) are the relations of $B(\infty, \infty, r)$ and the relations (Q_4) and (Q_5) describe the C_∞^e action on $B(\infty, \infty, r)$. Therefore we obtain the following.

Corollary 3.3. *The homomorphism $\psi: B(de, e, r) \rightarrow C_\infty^e \ltimes B(\infty, \infty, r)$ given by $\psi(z) = z$, $\psi(t_i) = t_i$ and $\psi(s_j) = s_j$ for $i \in \mathbb{Z}$ and $3 \leq j \leq r$ is an isomorphism.*

We propose the diagram shown in Figure 9(b) for the new presentation of $B(de, e, r)$. The diagram looks like the diagram for $B(\infty, \infty, r)$ in Figure 8(b). The action of z is illustrated by a curved arrow labeled e , describing the relation $z t_i = t_{i-e} z$.

3.2.2. Reflection group $G(de, e, r)$. As long as $d, d' \geq 2$, $B(de, e, r) \cong B(d'e, e, r)$. The parameter d only makes an appearance when it comes to the reflection group $G(de, e, r)$. As described

in [BMR98], adding the relations $z^d = 1$ and $a^2 = 1$ for all the other generators a to the Broué-Malle-Rouquier presentation of $B(de, e, r)$ gives a presentation for the complex reflection group $G(de, e, r)$. The generators are all reflections, of order 2 except z which is of order d .

The generators of the new presentation of $B(de, e, r)$ are those of the Broué-Malle-Rouquier presentation together with some conjugates of them. Thus, as it is the case in [BMR98], adding the relations $z^d = 1$ and $a^2 = 1$ for all the other generators a to the new presentation of $B(de, e, r)$ gives rise to a new presentation for the reflection group $G(de, e, r)$, where the generators are all reflections. This is a presentation on an infinite set of generators for a finite group! In fact, since $z^d = 1$, we have

$$t_{i+de} = z^d t_{i+de} = t_i z^d = t_i \quad \text{for all } i \in \mathbb{Z}.$$

Thus we have the following isomorphism.

Corollary 3.4. *The reflection group $G(de, e, r)$ for $d, r \geq 2$ and $e \geq 1$ is isomorphic to the semidirect product $C_d^e \ltimes G(de, de, r)$, where $C_d^e = \langle z \rangle$ is a cyclic group of order d . Hence $G(de, e, r)$ has the following presentation:*

- *Generators:* $\{z\} \cup T_{de} \cup S$ where $T_{de} = \{t_i \mid i \in \mathbb{Z}/de\}$ and $S = \{s_j \mid 3 \leq j \leq r\}$;
- *Relations:* all the relations of $G(de, de, r)$ in Theorem 2.1, along with
 - the relations $zt_i = t_{i-e}z$ and $zs_j = s_jz$ for $i \in \mathbb{Z}/de$ and $3 \leq j \leq r$ describing the semidirect product action,
 - the relations $z^d = 1$, $t_i^2 = 1$ and $s_j^2 = 1$ for $i \in \mathbb{Z}/de$ and $3 \leq j \leq r$ describing the order of the generating reflections.

In this presentation of $G(de, e, r)$, the generators can be represented by the following $r \times r$ matrices:

$$\overline{t_i} = \left(\begin{array}{cc|c} 0 & \zeta_{de}^{-i} & 0 \\ \zeta_{de}^i & 0 & 0 \\ \hline 0 & & I_{r-2} \end{array} \right), \quad \begin{aligned} \overline{z} &= \text{Diag}(\zeta_{de}^e, 1, 1, \dots, 1), \\ \overline{s_j} &= \text{permutation matrix of } (j-1 \ j), \end{aligned}$$

where ζ_{de} is a primitive de -th root of unity.

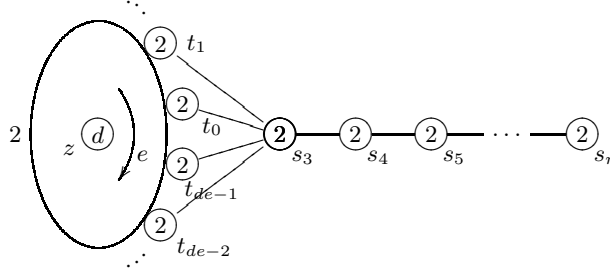
A diagram for the presentation of $G(de, e, r)$ is in Figure 10. It is obtained from the diagram for $B(de, e, r)$ in Figure 9(b) by identifying the node t_i with t_{i+de} for each $i \in \mathbb{Z}$. In particular, the disc at the left has de nodes on it, and the action of z twists this disc by e nodes. The numbers inside the nodes denote the orders of the generators (z has order d , all the others have order 2).

3.2.3. Maps between the groups $B(de, e, r)$ for different values of e . Denote by ι_e the natural embedding:

$$\iota_e : B(\infty, \infty, r) \hookrightarrow C_\infty^e \ltimes B(\infty, \infty, r) \cong B(de, e, r).$$

Once again, consider a sequence of natural numbers e_i such that $e_0 = 1$ and e_i divides e_{i+1} for each $i \geq 0$. Then there are embeddings $C_\infty^{e_{i+1}} \hookrightarrow C_\infty^{e_i}$ which maps the generator of $C_\infty^{e_{i+1}}$ to the $\frac{e_{i+1}}{e_i}$ -th power of the generator of $C_\infty^{e_i}$. These embeddings may be extended to $\iota_{e_i}^{e_{i+1}} : C_\infty^{e_{i+1}} \ltimes B(\infty, \infty, r) \hookrightarrow C_\infty^{e_i} \ltimes B(\infty, \infty, r)$. By Corollary 3.3, this map is thus an embedding between the braid groups:

$$\iota_{e_i}^{e_{i+1}} : B(de_{i+1}, e_{i+1}, r) \hookrightarrow B(de_i, e_i, r).$$

FIGURE 10. Diagram for the new presentation of $G(de, e, r)$

Hence we have the following commutative diagram, where the rows are exact.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & B(\infty, \infty, r) & \xrightarrow{\iota_{e_{i+1}}} & B(de_{i+1}, e_{i+1}, r) & \longrightarrow & C_{\infty}^{e_{i+1}} \longrightarrow 0 \\
 & & \parallel & & \downarrow \iota_{e_i}^{e_{i+1}} & & \downarrow \\
 0 & \longrightarrow & B(\infty, \infty, r) & \xrightarrow{\iota_{e_i}} & B(de_i, e_i, r) & \longrightarrow & C_{\infty}^{e_i} \longrightarrow 0
 \end{array}$$

Note that $\iota_{e_i}^{e_{i+1}} \circ \iota_{e_{i+1}} = \iota_{e_i}$ and $\iota_{e_i}^{e_{i+1}} \circ \iota_{e_{i+1}}^{e_{i+2}} = \iota_{e_i}^{e_{i+2}}$ for all $i \geq 0$, and that $B(\infty, \infty, r)$ is the inverse limit of the sequence

$$\cdots \hookrightarrow B(de_{i+1}, e_{i+1}, r) \xrightarrow{\iota_{e_i}^{e_{i+1}}} B(de_i, e_i, r) \hookrightarrow \cdots \xrightarrow{\iota_{e_1}^{e_2}} B(de_1, e_1, r) \xrightarrow{\iota_1^{e_1}} B(d, 1, r).$$

We remark that $B(d, 1, r) \cong B(\mathbf{B}_r)$, the braid group of type \mathbf{B}_r . This is discussed in greater length in §4.1.

3.2.4. Proof of Theorem 3.2. Similarly to the proof for the new presentation of $B(\infty, \infty, r)$, we add new generators $\{t_i \mid i \in \mathbb{Z} \setminus \{0, 1\}\}$ to the Broué-Malle-Rouquier presentation along with the relation $(Q_1) \ t_i t_{i-1} = t_j t_{j-1}$ for all $i, j \in \mathbb{Z}$. Then, from the proof of Theorem 3.1, we know that the relations $(R_4) + (R_5) + (R_6)$ are equivalent to $(Q_2) + (Q_3)$. The relation (R_3) is identical to (Q_5) . Therefore $B(de, e, r)$ has the following presentation.

- Generators: $\{z\} \cup T \cup S$ where $T = \{t_i \mid i \in \mathbb{Z}\}$ and $S = \{s_j \mid 3 \leq j \leq r\}$;
- Relations: the usual braid relations on S , along with
 - $(Q_1) \ t_i t_{i-1} = t_j t_{j-1} \text{ for all } i, j \in \mathbb{Z},$
 - $(Q_2) \ s_3 t_i s_3 = t_i s_3 t_i \text{ for all } i \in \mathbb{Z},$
 - $(Q_3) \ s_j t_i = t_i s_j \text{ for all } i \in \mathbb{Z} \text{ and } 4 \leq j \leq r,$
 - $(R_1) \ z t_1 t_0 = t_1 t_0 z,$
 - $(R_2) \ z \langle t_1 t_0 \rangle^e = t_0 z \langle t_1 t_0 \rangle^{e-1},$
 - $(Q_5) \ z s_j = s_j z \text{ for } 3 \leq j \leq r.$

The following claim completes the proof.

Claim. Assuming (Q_1) , the relation (Q_4) is equivalent to $(R_1) + (R_2)$.

Proof of Claim. Suppose that (Q_4) holds, i.e., $z t_i = t_{i-e} z$ for all $i \in \mathbb{Z}$. Then (R_1) holds because

$$z t_1 t_0 \stackrel{Q_4}{=} t_{1-e} z t_0 \stackrel{Q_4}{=} t_{1-e} t_{-e} z \stackrel{Q_1}{=} t_1 t_0 z.$$

When $e = 2m$,

$$\begin{aligned} z\langle t_1 t_0 \rangle^e &= z\langle t_1 t_0 \rangle^{2m} = z(t_1 t_0)^m \stackrel{Q_1}{=} z(t_{2m} t_{2m-1}) \cdots (t_2 t_1) \\ &\stackrel{Q_4}{=} t_0 z(t_{2m-1} t_{2m-2}) \cdots (t_3 t_2) t_1 \stackrel{Q_1}{=} t_0 z(t_1 t_0)^{m-1} t_1 = t_0 z\langle t_1 t_0 \rangle^{e-1}. \end{aligned}$$

When $e = 2m + 1$,

$$\begin{aligned} z\langle t_1 t_0 \rangle^e &= z\langle t_1 t_0 \rangle^{2m+1} = z(t_1 t_0)^m t_1 \stackrel{Q_1}{=} z(t_{2m+1} t_{2m}) \cdots (t_3 t_2) t_1 \\ &\stackrel{Q_4}{=} t_0 z(t_{2m} t_{2m-1}) \cdots (t_2 t_1) \stackrel{Q_1}{=} t_0 z(t_1 t_0)^m = t_0 z\langle t_1 t_0 \rangle^{e-1}. \end{aligned}$$

Therefore (R_2) holds.

Conversely, suppose that both (R_1) and (R_2) hold. First, we will show that (Q_4) holds for $i = e$, that is, $zt_e = t_0 z$. When $e = 2m$,

$$\begin{aligned} z\langle t_1 t_0 \rangle^e &= z(t_1 t_0)^m \stackrel{Q_1}{=} z(t_{2m} t_{2m-1}) \cdots (t_2 t_1) \quad \text{and} \\ t_0 z\langle t_1 t_0 \rangle^{e-1} &= t_0 z(t_1 t_0)^{m-1} t_1 \stackrel{Q_1}{=} t_0 z(t_{2m-1} t_{2m-2}) \cdots (t_3 t_2) t_1. \end{aligned}$$

Since $z\langle t_1 t_0 \rangle^e = t_0 z\langle t_1 t_0 \rangle^{e-1}$ by (R_2) , we have

$$zt_{2m}(t_{2m-1} \cdots t_1) = t_0 z(t_{2m-1} \cdots t_1).$$

Hence $zt_{2m} = t_0 z$, that is, $zt_e = t_0 z$. Therefore (Q_4) holds for $i = e$ when e is even.

When $e = 2m + 1$,

$$\begin{aligned} z\langle t_1 t_0 \rangle^e &= z(t_1 t_0)^m t_1 \stackrel{Q_1}{=} z(t_{2m+1} t_{2m}) \cdots (t_3 t_2) t_1 \quad \text{and} \\ t_0 z\langle t_1 t_0 \rangle^{e-1} &= t_0 z(t_1 t_0)^m \stackrel{Q_1}{=} t_0 z(t_{2m} t_{2m-1}) \cdots (t_2 t_1). \end{aligned}$$

Since $z\langle t_1 t_0 \rangle^e = t_0 z\langle t_1 t_0 \rangle^{e-1}$ by (R_2) , we have

$$zt_{2m+1}(t_{2m} \cdots t_1) = t_0 z(t_{2m} \cdots t_1).$$

Hence $zt_{2m+1} = t_0 z$, that is, $zt_e = t_0 z$. Therefore (Q_4) holds for $i = e$ when e is odd.

Now we will show that if (Q_4) holds for $i = k$, which we denote by $(Q_{4,k})$, then (Q_4) holds for $i = k - 1$ and $i = k + 1$. Assume that (Q_4) holds for $i = k$. Because

$$zt_{k+1} t_k \stackrel{Q_1}{=} zt_1 t_0 \stackrel{R_1}{=} t_1 t_0 z \stackrel{Q_1}{=} t_{k+1-e} t_{k-e} z \stackrel{Q_{4,k}}{=} t_{k+1-e} z t_k,$$

we have $zt_{k+1} = t_{k+1-e} z$, hence (Q_4) holds for $i = k + 1$. Similarly, because

$$t_{k-e} t_{k-e-1} z \stackrel{Q_1}{=} t_1 t_0 z \stackrel{R_1}{=} zt_1 t_0 \stackrel{Q_1}{=} zt_k t_{k-1} \stackrel{Q_{4,k}}{=} t_{k-e} z t_{k-1},$$

we have $t_{k-e-1} z = zt_{k-1}$, hence (Q_4) holds for $i = k - 1$. By induction on i , we conclude that (Q_4) holds for all $i \in \mathbb{Z}$. \square

3.3. Garside structures on $B(\infty, \infty, r)$ and $B(de, e, r)$. In this subsection, we show that the new presentations of $B(\infty, \infty, r)$ and $B(de, e, r)$ give rise to quasi-Garside structures.

Garside structures were defined by Dehornoy and Paris [DP99], in which the strategy and results of Garside [Gar69], Deligne [Del72], Brieskorn and Saito [BS72] still hold. A Garside structure provides tools for calculating in the group, for solving word and conjugacy problems, as well as for giving certain information about the group (such as being torsion-free). For a detailed description, see [DP99, DDGKM14]. We use the definition in [Dig06].

Definition 3.5. A monoid M is said to be *quasi-Garside* if the following conditions are satisfied:

- (i) M is *atomic*—that is, for every $m \in M$, the number of factors in a product equal to m is bounded;
- (ii) M is left- and right-cancellative;
- (iii) M is a lattice with respect to each of the orders defined by left divisibility and by right divisibility;
- (iv) M has a *Garside element* Δ for which the set of left divisors equals the set of right divisors, and this set generates M .

A quasi-Garside monoid satisfies Ore’s conditions [CP61], and thus embeds in its group of fractions.

Definition 3.6. Let M be a quasi-Garside monoid with Garside element Δ , and let G be the group of fractions of M . We identify the elements of M and their images in G . The pair (M, Δ) is called a *quasi-Garside structure* on G , and the triple (G, M, Δ) or just simply G is called a *quasi-Garside group*. The quasi-Garside monoid M of G is often denoted by G^+ .

When the set of left divisors of the Garside element Δ is finite, the word ‘quasi’ may be dropped for quasi-Garside.

If the monoid M is defined by a (positive) presentation with *homogeneous* relations—that is, for every relation, the left and right hand sides have equal length in the generators—then the first condition of being atomic is immediately satisfied, with the bound for the number of factors in a product equal to m being precisely the number of generators in an expression for m (since this number is the same for all expressions for m). This is always the case in the presentations we consider here.

For the second and third conditions of being cancellative and being a lattice, we introduce the notions of complementedness and completeness of Dehornoy [Deh03]. Let M be a monoid defined by a positive presentation $\langle S \mid R \rangle$. Let S^* denote the free monoid generated by S , and let ε denote the empty word.

Definition 3.7. For words w, w' on $S \cup S^{-1}$, we say that w *right-reverses* to w' , denoted $w \curvearrowright_r w'$, if w' is obtained from w (iteratively)

- either by deleting some subword $u^{-1}u$ for $u \in S^* \setminus \{\varepsilon\}$,
- or by replacing some subword $u^{-1}v$ for $u, v \in S^* \setminus \{\varepsilon\}$ with a word $v'u'^{-1}$ such that $uv' = vu'$ is a relation of R .

For any $u, v \in S^*$, $u^{-1}v \curvearrowright_r \varepsilon$ implies that $u = v$ in M .

Definition 3.8. The presentation $\langle S \mid R \rangle$ of M is said to be

- (i) *right-complemented* if for any $x, y \in S$, R has at most one relation of the form $x \cdots = y \cdots$ and no relation of the form $x \cdots = x \cdots$;
- (ii) *right-complete* if for any $u, v \in S^*$, $u = v$ in M implies $u^{-1}v \curvearrowright_r \varepsilon$.

The left versions of the above notions are defined symmetrically. In this subsection, several notions have a left and a right version. Without ‘left’ or ‘right’, we assume both versions. For instance, “ M is cancellative” means “ M is left- and right-cancellative”.

Let $B^+(e, e, r)$, $B^+(\infty, \infty, r)$, $B^+(de, e, r)$ and $B^+(\mathbf{B}_r)$ be the monoids defined by the presentations in Theorems 2.1, 3.1 and 3.2 and the presentation in (1) on page 6, respectively. It is known that $B^+(e, e, r)$ and $B^+(\mathbf{B}_r)$ are Garside. In the remaining of this subsection, we will show that $B^+(\infty, \infty, r)$ and $B^+(de, e, r)$ are quasi-Garside.

Remark 3.9. The Broué-Malle-Rouquier presentation for $B(de, e, r)$ does not give rise to a quasi-Garside structure for all $e \geq 1$. (Notice that when $e = 1$, the group itself is a Garside group because $B(d, 1, r) \cong B(\mathbf{B}_r)$.) Assume that the monoid defined by the presentation is quasi-Garside. Both $s_3 t_1 s_3 = t_1 s_3 t_1$ and $s_3 t_1 t_0 s_3 t_1 t_0 = t_1 t_0 s_3 t_1 t_0 s_3$ are common right multiples of s_3 and t_1 , hence

$$(s_3 t_1 s_3) \wedge (s_3 t_1 t_0 s_3 t_1 t_0) = (s_3 t_1)(s_3 \wedge t_0 s_3 t_1 t_0) = s_3 t_1,$$

is also a common right multiple of s_3 and t_1 , where \wedge denotes the left gcd. However $s_3 t_1$ is not a right multiple of t_1 , which is a contradiction. The same argument shows that Shi’s presentation for $B(\infty, \infty, r)$ does not give rise to a quasi-Garside structure.

3.3.1. *Garside structure on $B(\infty, \infty, r)$.* Since the presentation of $B^+(\infty, \infty, r)$ is homogeneous, the monoid is atomic. The conditions of being cancellative and a lattice can be checked by using complementedness and completeness.

Lemma 3.10. [Deh03, Corollary 6.2 and Propositions 3.3, 6.7 and 6.10] *Let M be a monoid defined by a complemented and complete presentation with S the set of generators. Then the following hold.*

- (i) *The monoid M is cancellative.*
- (ii) *Suppose that there exists S' such that $S \subseteq S' \subseteq S^*$ and for any $u, v \in S'$ there exist $u', v' \in S'$ with $uv' = vu'$ in M . Then M admits right lcm’s.*
- (iii) *Suppose that there exists S'' such that $S \subseteq S'' \subseteq S^*$ and for any $u, v \in S''$ there exist $u'', v'' \in S''$ with $v''u = u''v$ in M . Then M admits left lcm’s.*

The presentation of $B^+(\infty, \infty, r)$ is complemented. As for completeness, the cases to be considered are identical with those for $B^+(e, e, r)$ (Figures 7 and 8 in [CP11]), hence it can be checked in a manner entirely analogous to that for $B^+(e, e, r)$ given in [CP11]. In that paper, the presentation of $B^+(e, e, r)$ was shown to be complete by using the cube condition on all triples of generators. Therefore we have the following corollary by Lemma 3.10.

Corollary 3.11. *The monoid $B^+(\infty, \infty, r)$ is cancellative.*

Next, we will find sets S' and S'' satisfying the conditions in Lemma 3.10. Define a map

$$\begin{aligned} \psi: B^+(\mathbf{B}_{r-1}) &\rightarrow B^+(\infty, \infty, r) \\ b_1 &\mapsto t_1 t_0 \\ b_i &\mapsto s_{i+1} \quad \text{for } 2 \leq i \leq r-1. \end{aligned}$$

It is easy to see that ψ is a well-defined monoid homomorphism because the defining relations in $B^+(\mathbf{B}_{r-1})$ can be realized by the relations in $B^+(\infty, \infty, r)$. For example, $t_1 t_0 s_3 t_1 t_0 s_3 = s_3 t_1 t_0 s_3 t_1 t_0$ holds in $B^+(\infty, \infty, r)$ (see the proof of Theorem 3.1), hence $\psi(b_1)\psi(b_2)\psi(b_1)\psi(b_2) = \psi(b_2)\psi(b_1)\psi(b_2)\psi(b_1)$ holds in $B^+(\infty, \infty, r)$. We remark that ψ is injective. To see this, consider the composition with the morphism from $B(\infty, \infty, r)$ to $B(\mathbf{A}_{r-1})$ which maps t_i to σ_1 for $i \in \mathbb{Z}$ and s_j to σ_{j-1} for $3 \leq j \leq r$. The composition is the well-known embedding of $B^+(\mathbf{B}_{r-1})$ into $B^+(\mathbf{A}_{r-1})$ which maps b_1 to σ_1^2 and b_i to σ_i for $2 \leq i \leq r-1$.

The classical Garside element of the braid group $B(\mathbf{B}_{r-1})$, denoted by $\Delta_{\mathbf{B}_{r-1}}$, is the lcm of the generators $\{b_1, b_2, \dots, b_{r-1}\}$. It is a central element of $B^+(\mathbf{B}_{r-1})$, written as

$$\Delta_{\mathbf{B}_{r-1}} = (b_{r-1}b_{r-2} \cdots b_1)^{r-1}.$$

Let $\Lambda \in B^+(\infty, \infty, r)$ be the image of $\Delta_{\mathbf{B}_{r-1}}$ under ψ . Then Λ has the factorization

$$\Lambda = \psi(\Delta_{\mathbf{B}_{r-1}}) = (At_1t_0)^{r-1},$$

where $A = s_r s_{r-1} \cdots s_3$.

Let $L(\Lambda)$ and $R(\Lambda)$ denote the sets of all left and right divisors of Λ , respectively.

Proposition 3.12. *The element Λ is a Garside element of $B^+(\infty, \infty, r)$. That is, $L(\Lambda) = R(\Lambda)$, and $L(\Lambda)$ generates $B^+(\infty, \infty, r)$.*

We give the proof of the above proposition in §3.3.3.

The set $L(\Lambda) = R(\Lambda)$ meets the needs of S' and S'' in Lemma 3.10. Therefore $B^+(\infty, \infty, r)$ admits lcm's. It is easy to see that if a cancellative monoid admits lcm's then it admits gcd's. (For example, see Lemma 2.23 in [DDGKM14]).

So far, we have shown that the monoid $B^+(\infty, \infty, r)$ satisfies all the conditions in the definition of quasi-Garside monoids.

Theorem 3.13. *The presentation for $B(\infty, \infty, r)$ in Theorem 3.1 gives rise to a quasi-Garside structure, where $B^+(\infty, \infty, r)$ is the quasi-Garside monoid and Λ is a Garside element.*

We will see in §4.1 that the braid group $B(\infty, \infty, r)$ is isomorphic to the affine braid group $B(\tilde{\mathbf{A}}_{r-1})$. In [Dig06], Digne proposed a dual presentation of $B(\tilde{\mathbf{A}}_{r-1})$ which gives a quasi-Garside structure. This is different from the quasi-Garside structure on $B(\tilde{\mathbf{A}}_{r-1}) \cong B(\infty, \infty, r)$ given in Theorem 3.13.

3.3.2. Garside structure on $B(de, e, r)$. Recall from Corollary 3.3 that $B(de, e, r) \cong C_\infty^e \ltimes B(\infty, \infty, r)$ where $C_\infty^e = \langle z \rangle$ is an infinite cycle group. From the presentation for $B(de, e, r)$ in Theorem 3.2, C_∞^e acts on $B^+(\infty, \infty, r)$ by $zt_i z^{-1} = t_{i-e}$ and $zs_j z^{-1} = s_j$ for $i \in \mathbb{Z}$ and $3 \leq j \leq r$.

From §3.3.1, $(B(\infty, \infty, r), B^+(\infty, \infty, r), \Lambda)$ is a quasi-Garside group. On the other hand, $(C_\infty^e, (C_\infty^e)^+, z)$ is a Garside group, where $(C_\infty^e)^+ = \{z^n \mid n \geq 0\}$.

Picantin [Pic01] showed that the crossed product of Garside monoids is a Garside monoid. For semidirect products which are a special case of crossed products, it was directly proved in [Lee07].

Lemma 3.14 ([Lee07, Theorem 4.1]). *Let (G, G^+, Δ_G) and (H, H^+, Δ_H) be quasi-Garside groups. If ρ is an action of G on H^+ and Δ_H is fixed under ρ , then $(G \ltimes_\rho H, G^+ \ltimes_\rho H^+, (\Delta_G, \Delta_H))$ is a quasi-Garside group.*

Theorem 4.1 in [Lee07] is indeed stated not for quasi-Garside groups but for Garside groups, but its proof does not use finiteness of divisors of Δ_G or Δ_H . Hence the above lemma is true.

Since $z\Lambda = \Lambda z$, the Garside element Λ of $B^+(\infty, \infty, r)$ is fixed under the action of C_∞^e . Applying Lemma 3.14, we have the following result.

Theorem 3.15. *The presentation for $B(de, e, r)$ in Theorem 3.2 gives rise to a quasi-Garside structure on the imprimitive braid group $B(de, e, r) \cong C_\infty^e \rtimes B(\infty, \infty, r)$, where $B^+(de, e, r) \cong (C_\infty^e)^+ \rtimes B^+(\infty, \infty, r)$ is the quasi-Garside monoid and $z\Lambda$ is a Garside element.*

Indeed, $z^p\Lambda^q$ is a Garside element for any positive exponents p and q . A better choice would be $z^{\frac{r}{e\wedge r}}\Lambda^{\frac{e}{e\wedge r}}$ because it is the generator of the center of $B(de, e, r)$. Similarly $z^r\Lambda^e$ is also a good choice. However, if $p/q \neq r/e$, then $z^p\Lambda^q$ does not have a central power.

3.3.3. Proof of Proposition 3.12. When $r = 2$, the assertion is obvious. Hence we assume $r \geq 3$.

The properties required for the element Λ to be a Garside element of $B^+(\infty, \infty, r)$ are largely inherited from the Garside element $\Delta_{\mathbf{B}_{r-1}}$ of $B^+(\mathbf{B}_{r-1})$.

Let “ \doteq ” denote the equivalence in each of the positive monoids $B^+(\mathbf{B}_{r-1})$ and $B^+(\infty, \infty, r)$.

Notice that $\tau : B(\infty, \infty, r) \rightarrow B(\infty, \infty, r)$, defined by $\tau(t_i) = t_{i+1}$ and $\tau(s_j) = s_j$ for $i \in \mathbb{Z}$ and $3 \leq j \leq r$, induces an automorphism of the monoid $B^+(\infty, \infty, r)$. Then for all $k \in \mathbb{Z}$

$$\tau^k(\Lambda) \doteq \Lambda$$

because $\tau^k(A) = A$ and $\tau^k(t_1 t_0) = t_{k+1} t_k \doteq t_1 t_0$.

Since $\Delta_{\mathbf{B}_{r-1}}$ is central in $B^+(\mathbf{B}_{r-1})$, $\Delta_{\mathbf{B}_{r-1}} b_i \doteq b_i \Delta_{\mathbf{B}_{r-1}}$ for all $1 \leq i \leq r-1$. Therefore

$$\begin{aligned} \Lambda t_1 t_0 &\doteq t_1 t_0 \Lambda, \\ \Lambda s_j &\doteq s_j \Lambda \quad \text{for } 3 \leq j \leq r. \end{aligned}$$

Lemma 3.16. $\Lambda \doteq (At_r)(At_{r-1}) \cdots (At_1)$.

Proof. When $r = 3$, the assertion is true because

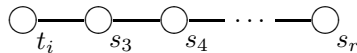
$$\Lambda = (s_3 t_1 t_0)^2 = s_3 t_1 t_0 s_3 t_1 t_0 \doteq s_3 t_3 t_2 s_3 t_2 t_1 \doteq s_3 t_3 s_3 t_2 s_3 t_1 = (At_3)(At_2)(At_1).$$

Hence we assume $r \geq 4$.

Notice that the following identities hold:

$$\begin{aligned} t_i(At_i) &\doteq (At_i)s_3 \quad \text{for } i \in \mathbb{Z}, \\ s_j(At_i) &\doteq (At_i)s_{j+1} \quad \text{for } i \in \mathbb{Z} \text{ and } 3 \leq j \leq r-1. \end{aligned}$$

One can prove it directly using the defining relations, or using the fact that $\{t_i, s_3, \dots, s_r\}$ for any $i \in \mathbb{Z}$ satisfies the braid relations described by the following diagram.



By moving t_k from the left to the right using the above identities, we have

$$(3) \quad t_k(At_k)(At_{k-1}) \cdots (At_2) \doteq (At_k)(At_{k-1}) \cdots (At_2)s_{k+1} \quad \text{for } 2 \leq k \leq r-1.$$

Now, we claim that the following identity holds.

$$(4) \quad (At_1 t_0)^k \doteq (At_{k+1})(At_k) \cdots (At_2)(s_{k+1} \cdots s_3)t_1 \quad \text{for } 1 \leq k \leq r-1.$$

The above equality is obvious for $k = 1$. Using induction on k , assume that (4) is true for some k with $1 \leq k \leq r - 2$. Then by (3)

$$\begin{aligned} (At_1 t_0)^{k+1} &= (At_1 t_0) (At_1 t_0)^k \\ &\doteq At_{k+2} t_{k+1} (At_{k+1}) (At_k) \cdots (At_2) (s_{k+1} \cdots s_3) t_1 \\ &\doteq At_{k+2} (At_{k+1}) (At_k) \cdots (At_2) s_{k+2} (s_{k+1} \cdots s_3) t_1. \end{aligned}$$

This shows that (4) is true.

Putting $k = r - 1$ to (4), we obtain $\Lambda \doteq (At_r)(At_{r-1}) \cdots (At_1)$. \square

Lemma 3.17. $\Lambda g \doteq \tau^r(g) \Lambda$ for all $g \in B^+(\infty, \infty, r)$.

Proof. Note that $\tau^r(t_i) = t_{i+r}$ for $i \in \mathbb{Z}$ and $\tau^r(s_j) = s_j$ for $3 \leq j \leq r$. Since $\Lambda s_j \doteq s_j \Lambda$ for $3 \leq j \leq r$, it suffices to show that $\Lambda t_i \doteq t_{i+r} \Lambda$ for all $i \in \mathbb{Z}$.

Since $\Lambda A \doteq A \Lambda$ and $\tau^k(\Lambda) \doteq \Lambda$ for $k \in \mathbb{Z}$,

$$(At_r) \Lambda \doteq (At_r) \tau^{-1}(\Lambda) \doteq (At_r) (At_{r-1}) \cdots (At_1) (At_0) \doteq \Lambda (At_0) \doteq A \Lambda t_0.$$

Hence $At_r \Lambda \doteq A \Lambda t_0$. Because $B^+(\infty, \infty, r)$ is cancellative,

$$t_r \Lambda \doteq \Lambda t_0.$$

Applying τ^i to the above identity, we have $t_{i+r} \Lambda \doteq \Lambda t_i$. \square

In [CP11, Proposition 3.5], the following was shown.

Lemma 3.18. *Let M be a cancellative monoid and let $h \in M$. If there is an automorphism ϕ of M such that $hg = \phi(g)h$ for all $g \in M$, then the set of left divisors of h is the same as the set of right divisors of h .*

Now, we are ready to show that Λ is a Garside element.

End of proof of Proposition 3.12. Since each b_i ($1 \leq i \leq r - 1$) is a left divisor of $\Delta_{\mathbf{B}_{r-1}}$ in $B^+(\mathbf{B}_{r-1})$, s_3, \dots, s_r and $t_1 t_0$ are left divisors of Λ in $B^+(\infty, \infty, r)$. Since each t_i ($i \in \mathbb{Z}$) is a left divisor of $t_1 t_0$, the set of left divisors of Λ contains $\{t_i \mid i \in \mathbb{Z}\} \cup \{s_3, \dots, s_r\}$, hence it generates $B^+(\infty, \infty, r)$. By Corollary 3.11 and Lemmas 3.17 and 3.18, the set of left divisors of Λ equals the set of right divisors of Λ . Consequently, Λ is a Garside element. \square

4. GEOMETRIC INTERPRETATION AND APPLICATIONS

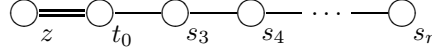
In this section, we explore some properties of $B(de, e, r)$ by using the interpretation of $B(de, e, r)$ as a geometric braid group.

4.1. Interpretation as geometric braids on $r + 1$ strings. In [BMR98], Broué, Malle and Rouquier constructed an isomorphism from $B(d, 1, r)$ to $B(\mathbf{B}_r)$ and an embedding of $B(de, e, r)$ into $B(\mathbf{B}_r)$. This subsection begins with reviewing them in our setting.

Consider the braid group $B(d, 1, r)$. By putting $e = 1$ to the relation $zt_i = t_{i-e}z$ of $B(de, e, r)$ in Theorem 3.2, we have $zt_i = t_{i-1}z$ for $i \in \mathbb{Z}$, hence for every $i \in \mathbb{Z}$

$$t_i = z^{-i} t_0 z^i.$$

Then it is straightforward to simplify the presentation of $B(d, 1, r)$ in Theorem 3.2 to the presentation illustrated by the diagram in Figure 11. Notice that it is the same as the diagram for $B(\mathbf{B}_r)$

FIGURE 11. Diagram for $B(d, 1, r)$

in Figure 3(b), where b_1 and b_2 are replaced with z and t_0 , respectively, and b_i is replaced with s_i for each $3 \leq i \leq r$. Therefore the groups $B(d, 1, r)$ and $B(\mathbf{B}_r)$ are isomorphic by the following map.

$$\begin{aligned}
 B(d, 1, r) &\xrightarrow{\cong} B(\mathbf{B}_r) \\
 z &\mapsto b_1 \\
 t_i &\mapsto b_1^{-i} b_2 b_1^i \quad (i \in \mathbb{Z}) \\
 s_j &\mapsto b_j \quad (3 \leq j \leq r)
 \end{aligned}$$

In §3.2.3, we saw that $B(\infty, \infty, r)$ is a subgroup of $B(d, 1, r)$, generated by $T \cup S$ where $T = \{t_i \mid i \in \mathbb{Z}\}$ and $S = \{s_j \mid 3 \leq j \leq r\}$. Similarly, using the map ι_1^e which embeds $B(de, e, r)$ into $B(d, 1, r)$ as a subgroup of index e by sending z (of $B(de, e, r)$) to z^e (of $B(d, 1, r)$), we have that $B(de, e, r)$ is isomorphic to a subgroup of $B(d, 1, r)$ generated by $\{z^e\} \cup T \cup S$. These embeddings can be summarized as follows.

$$\begin{array}{ccccccc}
 B(\infty, \infty, r) & \hookrightarrow & B(de, e, r) & \hookrightarrow & B(\mathbf{B}_r) & \hookrightarrow & B_{r+1} \\
 & & z & \mapsto & b_1^e & \mapsto & \sigma_1^{2e} \\
 t_i & \mapsto & t_i & \mapsto & b_1^{-i} b_2 b_1^i & \mapsto & \sigma_1^{-2i} \sigma_2 \sigma_1^{2i} \quad (i \in \mathbb{Z}) \\
 s_j & \mapsto & s_j & \mapsto & b_j & \mapsto & \sigma_j \quad (3 \leq j \leq r)
 \end{array}$$

Proposition 4.1. (i) *The braid group $B(de, e, r)$ is isomorphic to the subgroup of $B(\mathbf{B}_r)$ of index e generated by $\{b_1^e\} \cup \{b_1^{-i} b_2 b_1^i \mid i \in \mathbb{Z}\} \cup \{b_j \mid 3 \leq j \leq r\}$.*
(ii) *The braid group $B(\infty, \infty, r)$ is isomorphic to the subgroup of $B(\mathbf{B}_r)$ generated by $\{b_1^{-i} b_2 b_1^i \mid i \in \mathbb{Z}\} \cup \{b_j \mid 3 \leq j \leq r\}$.*

The first statement of the above proposition is Proposition 3.8 in [BMR98]. The following is a direct consequence of the fact that $B(de, e, r)$ is a finite index subgroup of $B(\mathbf{B}_r)$ and that $B(\mathbf{B}_r)$ is a Garside group.

Corollary 4.2. *$B(de, e, r)$ has a finite $K(\pi, 1)$, and is biautomatic. In particular, the word and conjugacy problems in $B(de, e, r)$ are solvable.*

Proof. Every Garside group is biautomatic [DP99] and has a finite $K(\pi, 1)$ [CMW04]. Any finite index subgroup of a biautomatic group is biautomatic, and the word and conjugacy problems are solvable in biautomatic groups [ECHLPT92]. It is obvious that any finite index subgroup of a group with a finite $K(\pi, 1)$ has a finite $K(\pi, 1)$. Since $B(\mathbf{B}_r)$ is a Garside group and $B(de, e, r)$ is a subgroup of $B(\mathbf{B}_r)$ of index e , we are done. \square

Recall that $B(d, 1, r) \cong B(\mathbf{B}_r) \cong B_{r+1,1}$ and that $\text{wd} : B_{r+1,1} \rightarrow \mathbb{Z}$ is defined by $\text{wd}(\sigma_1^2) = 1$ and $\text{wd}(\sigma_j) = 0$ for $2 \leq j \leq r$. Under the identification $B(d, 1, r) \cong B_{r+1,1}$, the homomorphism

$B(d, 1, r) \rightarrow C_\infty^1 \cong \mathbb{Z}$ in the exact sequence in §3.2.3 is the same as the winding number $\text{wd} : B_{r+1,1} \rightarrow \mathbb{Z}$. Hence we have the following commutative diagram, where the rows are exact.

$$\begin{array}{ccccccc} 0 & \longrightarrow & B(\infty, \infty, r) & \longrightarrow & B(d, 1, r) & \longrightarrow & C_\infty^1 \longrightarrow 0 \\ & & & & \downarrow \cong & & \downarrow \cong \\ 0 & \longrightarrow & \ker(\text{wd}) & \longrightarrow & B_{r+1,1} & \xrightarrow{\text{wd}} & \mathbb{Z} \longrightarrow 0 \end{array}$$

Then it follows that $B(\infty, \infty, r)$ is isomorphic to $\ker(\text{wd}) = \{g \in B_{r+1,1} \mid \text{wd}(g) = 0\}$.

Let $\text{wd}_e : B_{r+1,1} \rightarrow \mathbb{Z}/e$ be the homomorphism defined by $\text{wd}_e(g) = \text{wd}(g) \bmod e$. Then $\ker(\text{wd}_e) = \{g \in B_{r+1,1} \mid \text{wd}(g) \equiv 0 \bmod e\}$ is a subgroup of $B_{r+1,1}$ of index e . Because the subgroup generated by $\{\sigma_1^{2e}\} \cup \{\sigma_1^{-2i} \sigma_2 \sigma_1^{2i} \mid i \in \mathbb{Z}\} \cup \{\sigma_j \mid 3 \leq j \leq r\}$ is also of index e in $B_{r+1,1}$ by Proposition 4.1 and because it is a subgroup of $\ker(\text{wd}_e)$, it must coincide with $\ker(\text{wd}_e)$. Therefore $B(de, e, r)$ is isomorphic to $\ker(\text{wd}_e)$.

From the above discussions, Proposition 4.1 can be translated into the context of braids on $r+1$ strings and winding numbers as follows.

Corollary 4.3. *We have the following isomorphisms.*

$$\begin{aligned} B(d, 1, r) &\cong B_{r+1,1}, \\ B(de, e, r) &\cong \{g \in B_{r+1,1} \mid \text{wd}(g) \equiv 0 \bmod e\}, \\ B(\infty, \infty, r) &\cong \{g \in B_{r+1,1} \mid \text{wd}(g) = 0\}. \end{aligned}$$

The isomorphism $B(d, 1, r) \cong B_{r+1,1}$ is given by

$$z \mapsto \sigma_1^2, \quad t_i \mapsto \sigma_1^{-2i} \sigma_2 \sigma_1^{2i} \quad \text{for } i \in \mathbb{Z}, \quad s_j \mapsto \sigma_j \quad \text{for } 3 \leq j \leq r.$$

In this way, elements of $B(de, e, r)$ and $B(\infty, \infty, r)$ may be *visualised* as geometric braids. Notice that the braid groups $B(\infty, \infty, r)$ and $B(\tilde{\mathbf{A}}_{r-1})$ are isomorphic to the same subgroup of $B_{r+1,1}$, hence we have the following.

Corollary 4.4. *The braid group $B(\infty, \infty, r)$ is isomorphic to the braid group $B(\tilde{\mathbf{A}}_{r-1})$.*

Similarly to the case of $B(e, e, r)$, define a map

$$\begin{aligned} \tau : B(de, e, r) &\rightarrow B(de, e, r) \\ z &\mapsto z \\ t_i &\mapsto t_{i+1} \quad (i \in \mathbb{Z}) \\ s_j &\mapsto s_j \quad (3 \leq j \leq r). \end{aligned}$$

Then τ is an automorphism of $B(de, e, r)$, and $\tau(g) = \sigma_1^{-2} g \sigma_1^2$ for all $g \in B(de, e, r)$ when $B(de, e, r)$ is viewed as a subgroup of $B_{r+1,1}$.

If $e = 1$, then τ is an inner automorphism, that is, $\tau(g) = z^{-1} g z$ for all $g \in B(d, 1, r)$. But this is not necessarily the case for $e \geq 2$.

Proposition 4.5. *The automorphism $\tau : B(de, e, r) \rightarrow B(de, e, r)$ is an inner automorphism if and only if r and e are relatively prime.*

Proof. Identify $B(de, e, r)$ with $\{g \in B_{r+1,1} \mid \text{wd}(g) \equiv 0 \bmod e\}$.

Suppose that τ is an inner automorphism of $B(de, e, r)$. Then there exists $x \in B_{r+1,1}$ with $\text{wd}(x) \equiv 0 \pmod{e}$ such that

$$x^{-1}gx = \tau(g) = \sigma_1^{-2}g\sigma_1^2$$

for all $g \in B(de, e, r)$. Therefore $x\sigma_1^{-2}$ commutes with all the elements in $B(de, e, r)$. Hence it commutes with $\sigma_1^{2e}, \sigma_2, \dots, \sigma_r$. It is known that if $h \in B_{r+1}$ commutes with σ_i^k for some $k \neq 0$, then h commutes with σ_i (see [FRZ96]). Therefore $x\sigma_1^{-2}$ commutes with σ_i for all $1 \leq i \leq r$, hence $x\sigma_1^{-2}$ belongs to the center of B_{r+1} , which is the infinite cyclic group generated by the full twist Δ^2 , where $\Delta = (\sigma_1)(\sigma_2\sigma_1) \cdots (\sigma_r\sigma_{r-1} \cdots \sigma_1)$. Hence $x\sigma_1^{-2} = \Delta^{2k}$ for some $k \in \mathbb{Z}$. So $x = \Delta^{2k}\sigma_1^2$. Then

$$\text{wd}(x) = \text{wd}(\Delta^{2k}) + \text{wd}(\sigma_1^2) = kr + 1 \equiv 0 \pmod{e}.$$

Therefore r and e are relatively prime.

Conversely, suppose that r and e are relatively prime. Then $kr + 1 \equiv 0 \pmod{e}$ for some $k \in \mathbb{Z}$. Let $x = \sigma_1^2\Delta^{2k} \in B_{r+1}$. Since x is 1-pure and $\text{wd}(x) = kr + 1 \equiv 0 \pmod{e}$, $x \in B(de, e, r)$. Since Δ^{2k} is central, $x^{-1}gx = \sigma_1^{-2}g\sigma_1^2 = \tau(g)$ for all $g \in B(de, e, r)$, hence τ is an inner automorphism. \square

The next proposition will be used in the study of discreteness of translation numbers (in §4.3) and classification of periodic elements (in §4.4).

Proposition 4.6. *The embedding $\iota_1^e : B(de, e, r) \rightarrow B(d, 1, r)$ induces a finite-to-one map on the sets of conjugacy classes. More precisely, for $g, h \in B(de, e, r)$, $\iota_1^e(g)$ and $\iota_1^e(h)$ are conjugate in $B(d, 1, r)$ if and only if g is conjugate to $\tau^k(h)$ in $B(de, e, r)$ for some $0 \leq k < e$.*

Proof. Using Corollary 4.3, we identify $B(d, 1, r)$ and $B(de, e, r)$ with $B_{r+1,1}$ and $\{g \in B_{r+1,1} \mid \text{wd}(g) \equiv 0 \pmod{e}\}$, respectively. Let $g, h \in B(de, e, r)$.

Suppose that g is conjugate to $\tau^k(h)$ in $B(de, e, r)$ for some $0 \leq k < e$. Since $\tau^k(h) = \sigma_1^{-2k}h\sigma_1^{2k}$ is conjugate to h in $B(d, 1, r)$, g and h are conjugate in $B(d, 1, r)$.

Conversely, suppose that g and h are conjugate in $B(d, 1, r)$. Then $h = x^{-1}gx$ for some $x \in B(d, 1, r) = B_{r+1,1}$. Let $\text{wd}(x) \equiv -k \pmod{e}$ for some $0 \leq k < e$. Let $y = x\sigma_1^{2k}$. Then $y \in B(de, e, r)$ as $\text{wd}(y) \equiv 0 \pmod{e}$, and

$$y^{-1}gy = \sigma_1^{-2k}x^{-1}gx\sigma_1^{2k} = \sigma_1^{-2k}h\sigma_1^{2k} = \tau^k(h).$$

Therefore g is conjugate to $\tau^k(h)$ in $B(de, e, r)$. \square

4.2. Uniqueness of roots up to conjugacy. The following are well-known results on the uniqueness of roots in braid groups.

- (i) (González-Meneses [Gon03]) Let g and h be elements of $B(\mathbf{A}_r)$ such that $g^k = h^k$ for some nonzero integer k . Then g and h are conjugate in $B(\mathbf{A}_r)$.
- (ii) (Bardakov [Bar92], Kim and Rolfsen [KR03]) Let g and h be pure braids in $B(\mathbf{A}_r)$ such that $g^k = h^k$ for some nonzero integer k . Then g and h are equal.
- (iii) (Lee and Lee [LL10]) Let G be one of the braid groups of types \mathbf{A}_r , \mathbf{B}_r , $\tilde{\mathbf{A}}_{r-1}$ and $\tilde{\mathbf{C}}_{r-1}$. If $g, h \in G$ are such that $g^k = h^k$ for some nonzero integer k , then g and h are conjugate in G .

The first of these was conjectured by Makanin [Mak71] in the early seventies, and proved by González-Meneses. The second was initially proved by Bardakov by combinatorial arguments, and it follows easily from the biorderability of pure braids by Kim and Rolfsen. The third result is a generalization of the other two and comes from the following theorem, by viewing the braid groups of types \mathbf{B}_r , $\tilde{\mathbf{A}}_{r-1}$ and $\tilde{\mathbf{C}}_{r-1}$ as subgroups of B_{r+1} consisting of partially pure braids.

Theorem 4.7 ([LL10]). *Let P be a subset of $\{1, \dots, r+1\}$ with $1 \in P$. Let g and h be P -pure $(r+1)$ -braids such that $g^k = h^k$ for some nonzero integer k . Then there exists a P -straight $(r+1)$ -braid x with $h = x^{-1}gx$ and $\text{wd}(x) = 0$.*

Applying Theorem 4.7 to 1-pure $(r+1)$ -braids, and using the isomorphisms in Corollary 4.3, we obtain the following uniqueness of roots up to conjugacy in $B(de, e, r)$ and $B(\infty, \infty, r)$.

Corollary 4.8. *Let $g, h \in B(de, e, r)$ be such that $g^k = h^k$ for some nonzero integer k . Then g and h are conjugate in $B(de, e, r)$. Furthermore, a conjugating element x can be chosen from the subgroup $B(\infty, \infty, r)$ so that $h = x^{-1}gx$.*

Corollary 4.9. *If $g, h \in B(\infty, \infty, r)$ are such that $g^k = h^k$ for some nonzero integer k , then g and h are conjugate in $B(\infty, \infty, r)$.*

Question. Does the uniqueness of roots up to conjugacy hold in $B(e, e, r)$ and in the braid groups of real reflection groups of types other than \mathbf{A}_r , \mathbf{B}_r , $\tilde{\mathbf{A}}_{r-1}$ and $\tilde{\mathbf{C}}_{r-1}$?

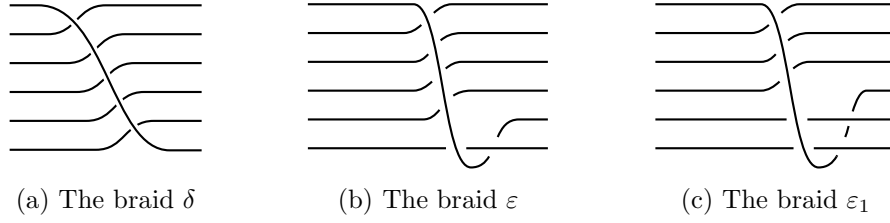
4.3. Discreteness of translation numbers. Translation numbers, introduced by Gersten and Short [GS91], are quite useful since it has both algebraic and geometric aspects. For a finitely generated group G and a finite set X of semigroup generators for G , the *translation number* of an element $g \in G$ with respect to X is defined by

$$t_{G,X}(g) = \liminf_{n \rightarrow \infty} \frac{|g^n|_X}{n},$$

where $|\cdot|_X$ denotes the minimal word-length in the alphabet X . When A is a set of group generators, $|g|_A$ and $t_{G,A}(g)$ indicate $|g|_{A \cup A^{-1}}$ and $t_{G,A \cup A^{-1}}(g)$, respectively. Kapovich [Kap97] and Conner [Con00] suggested the following notions: a finitely generated group G is said to be

- (i) *translation separable* if for some (and hence for any) finite set X of semigroup generators for G the translation numbers of non-torsion elements are strictly positive;
- (ii) *translation discrete* if it is translation separable and for some (and hence for any) finite set X of semigroup generators for G the set $t_{G,X}(G)$ has 0 as an isolated point;
- (iii) *strongly translation discrete* if it is translation separable and for some (and hence for any) finite set X of semigroup generators for G and for any real number r the number of conjugacy classes $[g] = \{h^{-1}gh : h \in G\}$ with $t_{G,X}(g) \leq r$ is finite. (The translation number is constant on each conjugacy class.)

There are several results on translation numbers in geometric and combinatorial groups. Biautomatic groups are translation separable [GS91]. Word hyperbolic groups are strongly translation discrete, and moreover, the translation numbers in a word hyperbolic group are rational with uniformly bounded denominators [Gro87, BGSS91, Swe95]. Artin groups of finite type are translation discrete [Bes99]. Garside groups are strongly translation discrete, and the translation numbers are rational with uniformly bounded denominators [CMW04, Lee07, LL07].

FIGURE 12. Braid pictures for δ , ε and ε_1 in B_6

Translation numbers of the braid groups $B(\infty, \infty, r)$ and $B(de, e, r)$ have the following properties.

Theorem 4.10. *The braid group $B(\infty, \infty, r)$ is translation discrete, and the braid group $B(de, e, r)$ is strongly translation discrete.*

Proof. It is known that a subgroup of a translation discrete group is translation discrete [Con98]. Since $B(d, 1, r) \cong B(\mathbf{B}_r)$ is strongly translation discrete [Lee07] and since $B(de, e, r)$ and $B(\infty, \infty, r)$ are subgroups of $B(d, 1, r)$, the groups $B(de, e, r)$ and $B(\infty, \infty, r)$ are translation discrete.

Let $G = B(d, 1, r)$ and $H = B(de, e, r)$. Identify G and H with $B_{r+1,1}$ and $\{g \in B_{r+1,1} \mid \text{wd}(g) \equiv 0 \pmod{e}\}$, respectively. Choose a finite set of generators, say X , for H . Then $Y = X \cup \{\sigma_1^2\}$ is a finite set of generators for G . Choose any real number r , and let

$$\begin{aligned} \mathcal{A} &= \{h \in H \mid t_{H,X}(h) \leq r\}, \\ \mathcal{B} &= \{g \in G \mid t_{G,Y}(g) \leq r\}. \end{aligned}$$

Notice that for any $h \in H$, $|h|_Y \leq |h|_X$, hence

$$t_{G,Y}(h) \leq t_{H,X}(h).$$

Therefore $\mathcal{A} \subset \mathcal{B}$. Because G is strongly translation discrete, there are finitely many conjugacy classes in \mathcal{B} . Hence there are finitely many conjugacy classes in \mathcal{A} by Proposition 4.6. \square

4.4. Classification of periodic elements. An element g in a braid group is said to be *periodic* if it has a central power. In this subsection, we classify periodic elements in $B(de, e, r)$. Here the group $B(de, e, r)$ is regarded as a subgroup of the braid group B_{r+1} .

4.4.1. Periodic elements in B_{r+1} and $B_{r+1,1}$. The center of the Artin braid group B_{r+1} is an infinite cyclic group generated by Δ^2 , where $\Delta = \sigma_1(\sigma_2\sigma_1) \dots (\sigma_r \dots \sigma_1)$. It is a classical theorem of Brouwer, Kerékjártó and Eilenberg [Bro19, Ker19, Eil34] that an $(r+1)$ -braid is periodic if and only if it is conjugate to a power of either δ or ε , where $\delta = \sigma_r \dots \sigma_1$ and $\varepsilon = \delta\sigma_1$. See Figure 12(a,b).

The center of $B_{r+1,1} \cong B(\mathbf{B}_r) \cong B(d, 1, r)$ is also the infinite cyclic group generated by Δ^2 , and every periodic element of $B(\mathbf{B}_r)$ is conjugate to a power of ε .

Similar statements hold for the braid groups of other finite types and the braid group $B(e, e, r)$. For example, see [LL11]. Bessis [Bes06b] explored many important properties of periodic elements in the context of braid groups of complex reflection groups.

4.4.2. *Periodic elements in $B(de, e, r)$.* The center of $B(de, e, r)$ is an infinite cyclic group generated by

$$\Delta_{(de, e, r)} = z^{\frac{r}{e \wedge r}} (At_1 t_0)^{\frac{e(r-1)}{e \wedge r}},$$

where $A = s_r \cdots s_3 \in B(de, e, r)$ [BMR98].

Lemma 4.11. *Let $A = s_r \cdots s_3 \in B(de, e, r)$. Then the following hold.*

- (i) $At_k At_{k-1} \cdots At_1 = \sigma_1^{-2k} \varepsilon^k$ for all $k \in \mathbb{Z}$.
- (ii) $(At_1 t_0)^{r-1} = \sigma_1^{-2r} \Delta^2$.
- (iii) $(At_1 t_0)^{r-1} = At_{j+r} \cdots At_{j+1}$ for all $j \in \mathbb{Z}$.

Proof. (i) Recall that $t_i = \sigma_1^{-2i} \sigma_2 \sigma_1^{2i}$ for all $i \in \mathbb{Z}$. Since for every $i \in \mathbb{Z}$

$$At_i = A \sigma_1^{-2i} \sigma_2 \sigma_1^{2i} = \sigma_1^{-2i} A \sigma_2 \sigma_1^{2(i-1)} = \sigma_1^{-2i} \varepsilon \sigma_1^{2(i-1)},$$

we have for every $k \in \mathbb{Z}$

$$At_k At_{k-1} \cdots At_1 = (\sigma_1^{-2k} \varepsilon \sigma_1^{2(k-1)}) (\sigma_1^{-2(k-1)} \varepsilon \sigma_1^{2(k-2)}) \cdots (\sigma_1^{-2} \varepsilon) = \sigma_1^{-2k} \varepsilon^k.$$

(ii) Let $\varepsilon_1 = (\sigma_r \cdots \sigma_2 \sigma_1) \sigma_1 \sigma_2 = A \sigma_2 \sigma_1^2 \sigma_2$. See Figure 12(c). Geometrically, ε_1 is the $1/(r-1)$ -twist around the first two strings. Hence ε_1^{r-1} is the full twist except that the first two strings are straight, that is,

$$\varepsilon_1^{r-1} = \sigma_1^{-2} \Delta^2.$$

Also notice that σ_1 commutes with ε_1 and A . Since

$$At_1 t_0 = A(\sigma_1^{-2} \sigma_2 \sigma_1^2) \sigma_2 = \sigma_1^{-2} A \sigma_2 \sigma_1^2 \sigma_2 = \sigma_1^{-2} \varepsilon_1,$$

we have

$$(At_1 t_0)^{r-1} = (\sigma_1^{-2} \varepsilon_1)^{r-1} = \sigma_1^{-2(r-1)} \varepsilon_1^{r-1} = \sigma_1^{-2(r-1)} \sigma_1^{-2} \Delta^2 = \sigma_1^{-2r} \Delta^2.$$

(iii) Setting $k = r$ to (i), we have

$$(5) \quad At_r \cdots At_1 = \sigma_1^{-2r} \varepsilon^r = \sigma_1^{-2r} \Delta^2.$$

Notice that $\tau(A) = A$, $\tau(t_k) = t_{k+1}$ and $\tau(t_1 t_0) = t_2 t_1 = t_1 t_0$. Applying τ^j to both sides of (5) and using (ii), we obtain (iii). \square

Recall $At_1 t_0 = \sigma_1^{-2} \varepsilon_1$ from the proof of Lemma 4.11. Then the generator $\Delta_{(de, e, r)}$ of the center of $B(de, e, r)$ is written as

$$\begin{aligned} \Delta_{(de, e, r)} &= z^{\frac{r}{e \wedge r}} (At_1 t_0)^{\frac{e(r-1)}{e \wedge r}} = (\sigma_1^{2e})^{\frac{r}{e \wedge r}} (\sigma_1^{-2} \varepsilon_1)^{\frac{e(r-1)}{e \wedge r}} \\ &= (\sigma_1^2)^{\frac{e}{e \wedge r}} (\varepsilon_1^{r-1})^{\frac{e}{e \wedge r}} = (\sigma_1^2)^{\frac{e}{e \wedge r}} (\sigma_1^{-2} \Delta^2)^{\frac{e}{e \wedge r}} = (\Delta^2)^{\frac{e}{e \wedge r}}. \end{aligned}$$

Therefore $\Delta_{(de, e, r)}$ is a power of Δ^2 . Note that $\text{wd}(\Delta_{(de, e, r)}) = \frac{er}{e \wedge r} = e \vee r$. In other words, the center of $B(de, e, r)$ is the intersection of $B(de, e, r)$ and the center of $B_{r+1, 1}$. Hence we have the following.

Proposition 4.12. *An element $g \in B(de, e, r)$ is periodic in $B(de, e, r)$ if and only if it is periodic in $B_{r+1, 1}$.*

Now we classify periodic elements in $B(de, e, r)$.

Definition 4.13. Set $\lambda = z A t_e A t_{e-1} \cdots A t_1 \in B(de, e, r)$.

Recall that $z = \sigma_1^{2e}$. By Lemma 4.11, $\lambda = \sigma_1^{2e}(\sigma_1^{-2e}\varepsilon^e) = \varepsilon^e$, hence λ is periodic in $B_{r+1,1}$. By Proposition 4.12, λ is periodic in $B(de, e, r)$. It follows also from

$$\lambda^{\frac{r}{\varepsilon \wedge r}} = \varepsilon^{\frac{er}{\varepsilon \wedge r}} = (\Delta^2)^{\frac{e}{\varepsilon \wedge r}} = \Delta_{(de, e, r)}.$$

Theorem 4.14. *In $B(de, e, r)$, an element g is periodic if and only if g is conjugate to a power of λ .*

Proof. First, notice that λ is conjugate to $\tau(\lambda)$ in $B(de, e, r)$ because

$$(At_1)\lambda(At_1)^{-1} = At_1 z At_e \cdots At_2 = z At_{e+1} At_e \cdots At_2 = \tau(\lambda).$$

Therefore λ^q is conjugate to $\tau^k(\lambda^q)$ in $B(de, e, r)$ for all $k, q \in \mathbb{Z}$.

Suppose that g is a periodic element in $B(de, e, r)$. Then g is periodic in $B_{r+1,1}$ by Proposition 4.12, hence it is conjugate to ε^p for some $p \in \mathbb{Z}$. Since $\text{wd}(g) = p \text{wd}(\varepsilon) = p \equiv 0 \pmod{e}$, $p = qe$ for some $q \in \mathbb{Z}$. Then g is conjugate to $\varepsilon^{qe} = \lambda^q$ in $B_{r+1,1}$. Hence g is conjugate to $\tau^k(\lambda^q)$ in $B(de, e, r)$ for some $0 \leq k < e$ by Proposition 4.6. Therefore g is conjugate to λ^q in $B(de, e, r)$.

The converse direction is obvious. \square

4.4.3. Comparison with the results of Bessis. Here we assume $d, e, r \geq 2$. We recall the results of Bessis [Bes06b] on periodic elements in the braid groups associated with *well-generated* complex reflection groups. The reflection group $G(de, e, r)$ is *not* well-generated. However we will see that Bessis' results hold for $B(de, e, r)$ except the existence of the dual Garside element δ .

Let G be a complex reflection group on V . Let $d_1 \leq d_2 \leq \cdots \leq d_r$ be degrees of G and $d_1^* \geq d_2^* \geq \cdots \geq d_r^*$ be codegrees of G . The largest degree d_r is called the *Coxeter number* of G , which we denote by h . An integer p is called a *regular number* if there exist an element $g \in G$ and a complex p -th root ζ of unity such that $\ker(g - \zeta) \cap V^{\text{reg}} \neq \emptyset$, where V^{reg} is the complement in V of the reflecting hyperplanes of G . The complex reflection group G is called *well-generated* if G can be generated by r reflections. It is known that G is well-generated if and only if G is a *duality group*, i.e., $d_i + d_i^* = d_r$ for all $1 \leq i \leq r$.

The following theorem collects Bessis' results on periodic elements in braid groups; see Lemma 6.11 and Theorems 1.9, 8.2, 12.3, 12.5 in [Bes06b]. The equivalence between (a) and (b) in the theorem was proved by Lehrer and Springer [LS99] and by Lehrer and Michel [LM03], and it does not require well-generatedness of the complex reflection group G .

Theorem 4.15 ([Bes06b]). *Let G be an irreducible well-generated complex reflection group, with degrees d_1, \dots, d_r , codegrees d_1^*, \dots, d_r^* and Coxeter number h . Then its braid group $B(G)$ admits the dual Garside structure with Garside element δ , and the following hold.*

- (i) *The element $\mu = \delta^h$ is central in $B(G)$ and lies in the pure braid group of G .*
- (ii) *Let $h' = h/(d_1 \wedge \cdots \wedge d_r)$. The center of $B(G)$ is a cyclic group generated by $\delta^{h'}$.*
- (iii) *Let p be a positive integer, and let*

$$A(p) = \{1 \leq i \leq r : p \mid d_i\} \quad \text{and} \quad B(p) = \{1 \leq i \leq r : p \mid d_i^*\}.$$

Then $|A(p)| \leq |B(p)|$, and the following conditions are equivalent:

- (a) $|A(p)| = |B(p)|$;
- (b) p is regular;
- (c) there exists a p -th root of μ .

Moreover, the p -th root of μ , if exists, is unique up to conjugacy in $B(G)$.

The Coxeter groups of types $\mathbf{A}_r, \mathbf{B}_r, \mathbf{D}_r, \mathbf{I}_2(e)$ and the complex reflection group $G(e, e, r)$ are all irreducible well-generated complex reflection groups. But $G(de, e, r)$ is *not* well-generated, hence the above theorem cannot be applied.

We remark that Bessis' results hold for $B(de, e, r)$ except the existence of δ . The degrees and codegrees of $G(de, e, r)$ are as follows [BMR98]:

$$\begin{aligned}\{d_1, \dots, d_r\} &= \{e, 2e, \dots, (r-1)e, r\} = \{e, 2e, \dots, (r-1)e\} \cup \{r\}, \\ \{d_1^*, \dots, d_r^*\} &= \{0, e, 2e, \dots, (r-1)e\} = \{e, 2e, \dots, (r-1)e\} \cup \{0\}.\end{aligned}$$

Therefore $d_1 \wedge \dots \wedge d_r = e \wedge r$, $h = e(r-1)$ and $h' = h/(d_1 \wedge \dots \wedge d_r) = e(r-1)/(e \wedge r)$.

- (i) Let $\mu = z^r (At_1 t_0)^{e(r-1)}$. Then μ is central in $B(de, e, r)$.
- (ii) The center of $B(de, e, r)$ is an infinite cyclic group generated by $\Delta_{(de, e, r)} = z^{\frac{r}{e \wedge r}} (At_1 t_0)^{\frac{e(r-1)}{e \wedge r}}$.
Notice that $(\Delta_{(de, e, r)})^{d_1 \wedge \dots \wedge d_r} = (\Delta_{(de, e, r)})^{e \wedge r} = \mu$.
- (iii) The following are equivalent.
 - (a) $|A(p)| = |B(p)|$;
 - (b) p is regular;
 - (c) there exists a p -th root of μ ;
 - (d) $p \mid r$.

Moreover, the p -th root of μ , if exists, is unique up to conjugacy in $B(de, e, r)$.

In the above, the equivalence between (c) and (d) follows from Theorem 4.14 because λ is an r -th root of μ . The other statements are immediate.

Question. Can we generalize the approach of Bessis in [Bes06b] to the braid group $B(de, e, r)$?

5. $\tilde{\mathbf{A}}$ -TYPE PRESENTATION FOR $B(de, e, r)$

The presentation of $B(de, e, r)$ in Theorem 3.2 describes the semidirect product decomposition $B(de, e, r) \cong C_\infty^e \ltimes B(\infty, \infty, r)$: the last two relations describe the action of z on $B(\infty, \infty, r)$, where z is the generator of the infinite cyclic group C_∞^e , and the others are the relations of $B(\infty, \infty, r)$.

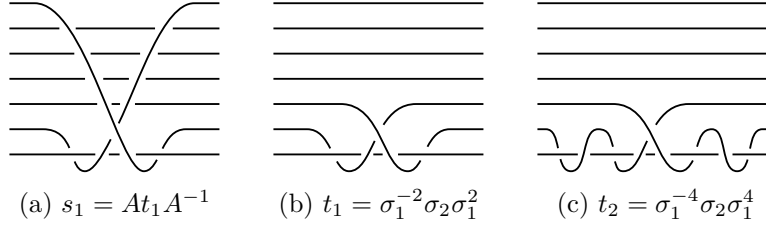
Because $B(\infty, \infty, r) \cong B(\tilde{\mathbf{A}}_{r-1})$, the group $B(de, e, r)$ has an $\tilde{\mathbf{A}}$ -type presentation with generators $\{s_1, s_2, \dots, s_r\}$ of $B(\tilde{\mathbf{A}}_{r-1})$ along with z . In this section, we explicitly compute this presentation.

Throughout this section, we assume $r \geq 3$ and regard the groups $B(\infty, \infty, r)$, $B(\tilde{\mathbf{A}}_{r-1})$ and $B(de, e, r)$ as subgroups of the braid group B_{r+1} , hence $B(\tilde{\mathbf{A}}_{r-1}) = B(\infty, \infty, r) \subset B(de, e, r) \subset B_{r+1}$. Let A and B denote the braids

$$A = \sigma_r \sigma_{r-1} \cdots \sigma_3, \quad B = A \sigma_2 = \sigma_r \sigma_{r-1} \cdots \sigma_3 \sigma_2.$$

Recall from §4.1 that the generators $z \in B(de, e, r)$ and $t_i, s_j \in B(\infty, \infty, r)$ are

$$\begin{aligned}z &= \sigma_1^{2e}, \\ t_i &= \sigma_1^{-2i} \sigma_2 \sigma_1^{2i} \quad \text{for } i \in \mathbb{Z}, \\ s_j &= \sigma_j \quad \text{for } 3 \leq j \leq r.\end{aligned}$$

FIGURE 13. Braid pictures for s_1 , t_1 and t_2 when $r = 6$

Recall from §2.3 that the generators s_1, \dots, s_r of $B(\tilde{\mathbf{A}}_{r-1})$ are

$$\begin{aligned} s_1 &= At_1A^{-1} = A\sigma_1^{-2}\sigma_2\sigma_1^2A^{-1}, \\ s_j &= \sigma_j \quad \text{for } 2 \leq j \leq r. \end{aligned}$$

Note that $A, B \in B(\tilde{\mathbf{A}}_{r-1})$. See Figure 13 for the braid pictures of s_1 , t_1 and t_2 .

Recall the diagrams for the usual (Artin) presentation of $B(\tilde{\mathbf{A}}_{r-1})$ and the new presentation of $B(\infty, \infty, r)$ shown in Figures 5 and 8(b). These diagrams give natural diagram automorphisms κ and τ , obtained respectively by rotating or by shifting by one node the diagram. The automorphism $\kappa : B(\tilde{\mathbf{A}}_{r-1}) \rightarrow B(\tilde{\mathbf{A}}_{r-1})$ is given by $s_j \mapsto s_{j+1 \bmod r}$ for $j \in \mathbb{Z}/r$. It is not hard to see from braid relations that κ is the conjugation by $\varepsilon = \sigma_r \cdots \sigma_2\sigma_1^2 = A\sigma_2\sigma_1^2 = B\sigma_1^2$, that is,

$$\kappa(g) = \varepsilon^{-1}g\varepsilon \quad \text{for } g \in B(\tilde{\mathbf{A}}_{r-1}).$$

The automorphism $\tau : B(\infty, \infty, r) \rightarrow B(\infty, \infty, r)$ is given by $t_i \mapsto t_{i+1}$ and $s_j \mapsto s_j$ for $i \in \mathbb{Z}$ and $3 \leq j \leq r$. It is the conjugation by σ_1^2 , that is,

$$\tau(g) = \sigma_1^{-2}g\sigma_1^2 \quad \text{for } g \in B(\infty, \infty, r).$$

Since $\varepsilon = B\sigma_1^2$, κ and τ are related by

$$\tau(g) = \kappa(BgB^{-1}) \quad \text{for } g \in B(\infty, \infty, r) = B(\tilde{\mathbf{A}}_{r-1}).$$

Lemma 5.1. *The braids s_1B and s_j for $3 \leq j \leq r$ commute with σ_1 , hence $\tau(s_1B) = s_1B$ and $\tau(s_j) = s_j$ for $3 \leq j \leq r$.*

Proof. Note that $s_1B = (A\sigma_1^{-2}\sigma_2\sigma_1^2A^{-1})(A\sigma_2) = A\sigma_1^{-2}\sigma_2\sigma_1^2\sigma_2 = \sigma_1^{-2}A\sigma_2\sigma_1^2\sigma_2 = \sigma_1^{-2}\varepsilon_1$, where the braid ε_1 is illustrated in Figure 12(c). Since ε_1 commutes with σ_1 , so does s_1B . It is obvious that σ_1 commutes with $s_j = \sigma_j$ for $3 \leq j \leq r$. \square

Lemma 5.2. *For any $k \geq 1$,*

$$\begin{aligned} \tau^k(s_2) &= \kappa(B)\kappa^2(B) \cdots \kappa^k(B)\kappa^k(A^{-1})\kappa^{k-1}(B^{-1}) \cdots \kappa(B^{-1}) \\ &= \langle s_1B \rangle^{k(r-1)} \left(\langle s_1B \rangle^{k(r-1)-1} \right)^{-1}. \end{aligned}$$

Proof. We prove the first equality by induction on k . The case $k = 1$ is true since

$$\tau(s_2) = \kappa(Bs_2B^{-1}) = \kappa(Bs_2(As_2)^{-1}) = \kappa(BA^{-1}) = \kappa(B)\kappa(A^{-1}).$$

Assume that the assertion is true for some $k \geq 1$. Then

$$\begin{aligned}\tau^{k+1}(s_2) &= \tau(\tau^k(s_2)) = \kappa(B\tau^k(s_2)B^{-1}) \\ &= \kappa(B \cdot \kappa(B) \cdots \kappa^k(B) \kappa^k(A^{-1}) \kappa^{k-1}(B^{-1}) \cdots \kappa(B^{-1}) \cdot B^{-1}) \\ &= \kappa(B) \kappa^2(B) \cdots \kappa^{k+1}(B) \kappa^{k+1}(A^{-1}) \kappa^k(B^{-1}) \cdots \kappa^2(B^{-1}) \kappa(B^{-1}).\end{aligned}$$

Therefore the assertion is true for $k+1$.

By a straightforward computation, one can easily see that $\kappa(B) \kappa^2(B) \cdots \kappa^k(B) = \langle s_1 B \rangle^{k(r-1)}$ and that $\kappa(B) \kappa^2(B) \cdots \kappa^{k-1}(B) \kappa^k(A) = \langle s_1 B \rangle^{k(r-1)-1}$. For example,

$$\begin{aligned}\kappa(B) \kappa^2(B) &= \kappa(s_r \cdots s_2) \kappa^2(s_r \cdots s_2) \\ &= (s_1 s_r s_{r-1} \cdots s_3)(s_2 s_1 s_r \cdots s_4) = \langle s_1 s_r \cdots s_2 \rangle^{2(r-1)} \\ &= \langle s_1 B \rangle^{2(r-1)}.\end{aligned}$$

Therefore the second equality is immediate. \square

The action of z on $B(\tilde{\mathbf{A}}_{r-1})$ is given by $z^{-1}gz = \tau^e(g)$ for $g \in B(\tilde{\mathbf{A}}_{r-1})$. Notice that $\{s_1 B, s_2, \dots, s_r\}$ generates $B(\tilde{\mathbf{A}}_{r-1})$. Using Lemmas 5.1 and 5.2, the action of z on $B(\tilde{\mathbf{A}}_{r-1})$ can be described as

$$\begin{aligned}z^{-1}s_i z &= \tau^e(s_i) = s_i \quad \text{for } 3 \leq i \leq r, \\ z^{-1}s_1 B z &= \tau^e(s_1 B) = s_1 B, \\ z^{-1}s_2 z &= \tau^e(s_2) = \langle s_1 B \rangle^{e(r-1)} \left(\langle s_1 B \rangle^{e(r-1)-1} \right)^{-1}.\end{aligned}$$

These are equivalent to

$$\begin{aligned}zs_i &= s_i z \quad \text{for } 3 \leq i \leq r, \\ zs_1 B &= s_1 B z, \\ z \langle s_1 B \rangle^{e(r-1)} &= s_2 z \langle s_1 B \rangle^{e(r-1)-1}.\end{aligned}$$

This allows us to obtain a positive homogeneous presentation for $B(de, e, r)$ in terms of the conventional generators of $\tilde{\mathbf{A}}_{r-1}$. (We note, however, that the following presentation does not give rise to a quasi-Garside structure.)

Theorem 5.3. *The group $B(de, e, r) \cong C_\infty^e \rtimes B(\tilde{\mathbf{A}}_{r-1})$ for $d \geq 2$, $e \geq 1$ and $r \geq 3$ has the following presentation:*

- *Generators:* $\{z\} \cup \{s_i \mid 1 \leq i \leq r\}$;
- *Relations:*
 - (A₁) $s_i s_j = s_j s_i \quad \text{for } i - j \not\equiv \pm 1 \pmod{r},$
 - (A₂) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{for } 1 \leq i \leq r,$
 - (A₃) $zs_i = s_i z \quad \text{for } 3 \leq i \leq r$
 - (A₄) $zs_1 B = s_1 B z,$
 - (A₅) $z \langle s_1 B \rangle^{e(r-1)} = s_2 z \langle s_1 B \rangle^{e(r-1)-1},$

where $B = s_r s_{r-1} \cdots s_2$. Furthermore, on adding the relations $z^d = 1$ and $s_i^2 = 1$ for all $1 \leq i \leq r$, a presentation for the reflection group $G(de, e, r)$ is obtained, where the generators are all reflections.

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